A Bayesian Theory of Games
A Bayesian Theory of Games

Iterative conjectures and determination of equilibrium

JIMMY TENG
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Preface

This book introduces a new games theory equilibrium concept and solution algorithm that provide a unified treatment for broad categories of games that are presently solved using the different equilibrium concepts of Nash equilibrium, sub-game perfect equilibrium, Bayesian Nash equilibrium and perfect Bayesian equilibrium.

The new method achieves consistency in equilibrium results that current games theory at times fails to, such as those between Perfect Bayesian Equilibrium and backward induction (sub-game Perfect Equilibrium). The new equilibrium concept is Bayesian equilibrium by iterative conjectures (BEIC) and its associated algorithm is the Bayesian iterative conjecture algorithm. BEIC requires players to make predictions on the strategies of other players using the Bayesian iterative conjecture algorithm. The Bayesian iterative conjectures algorithm makes predictions starting from first order uninformative predictive distribution functions (or conjectures) and keeps updating with the Bayesian statistical decision theoretic and game theoretic reasoning until a convergence of conjectures is achieved. Information known by the players such as the reaction functions are thereby incorporated into the higher order conjectures and help to determine the convergent conjectures and the associated equilibrium.
In a BEIC, conjectures are consistent with the equilibrium or equilibriums they support and so rationality is achieved for actions, strategies and conjectures and (statistical) decision rule.

The BEIC approach is capable of analyzing a larger set of games than current games theory, including games with noisy inaccurate observations and games with multiple-sided incomplete information games. On the other hand, for the set of games analyzed by the current games theory, it generates smaller numbers of equilibriums and normally achieves uniqueness in equilibrium. It treats games with complete and perfect information as special cases of games with incomplete information and noisy observations, whereby the variance of the prior distribution function on type and the variance of the observation noise term tend to zero. Consequently, there is the issue of indeterminacy in statistical inference and decision-making in these games as the equilibrium solution depends on which variance tends to zero first. It therefore identifies equilibriums in these games that have so far eluded current treatments.

I thank the students of my 2005, 2008 and 2009 games theory classes (at the Department and Graduate Institute of Political Science in the National Taiwan University in Taipei, Taiwan) for their enthusiasm in learning, and interesting questions raised in class.

I thank the participants of my three-day games theory workshop (at the Graduate Institute of Political Science in the National Sun Yat Sen University in Kaohsiung, Taiwan) for their questions.

I thank the students of my 2011, 2012 and 2013 microeconomic theory and advanced microeconomics classes (at the School of Economics in the University of Nottingham Malaysia Campus) for their questions.
About the author

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Introduction

‘If a man will begin with certainties, he shall end in doubts; but if he will content to begin with doubts, he shall end in certainties.’

Francis Bacon (1561–1626).

There is a lot of criticism against Nash equilibrium (and its myriad refinements). There are many alternative equilibrium concepts being proposed. Yet, despite the criticism and the alternative equilibrium concepts, Nash equilibrium is still the dominant equilibrium concept. In terms of popularity of usage, no alternative equilibrium concept comes close. The most important reason is that Nash equilibrium is relatively easy to compute, while many of the alternative equilibrium concepts are algorithmically difficult to compute.

This book proposes a new equilibrium concept that overcomes many of the shortcomings of Nash equilibrium. The proposed new equilibrium concept also has the nice algorithmic property of easy computation. Consequently, not only could it solve games that the current approach based on Nash equilibrium could solve, it could also solve games that the current approach based on Nash equilibrium is unable to solve. The new equilibrium concept is Bayesian equilibrium by iterative conjectures.

Current Nash equilibrium-based games theory solves a game by asking which combinations of strategies constitute
equilibriums. The implicit assumption is that agents know the strategies adopted by the other agents and which equilibrium they are in, for otherwise they will not be able to react specifically to the optimal strategies of other agents but must react to the strategies of the other agents they predicted or expected or conjectured.\(^1\) This implicit assumption reduces the uncertainty facing the agents and simplifies computation and gives Nash equilibrium its strong appeal. Further refinements such as sub-game perfect equilibrium, Bayesian Nash equilibrium and Perfect Bayesian (Nash) equilibrium, though adding further requirements, do not change this implicit assumption.\(^2\)

Bayesian equilibrium by the iterative conjectures (BEIC) approach, in contrast, solves a game by assuming that the agents do not know the strategies adopted or will be adopted by other agents, and have no idea which equilibrium they are in or will be in. Therefore, to select a strategy they need to form predictions or expectations or conjectures about the strategies adopted or will be adopted by other agents and the equilibrium they are in or will be in, as well as conjectures about such conjectures, ad infinitum. They do so by starting with first order uninformative predictive probability distribution functions (or expectations or conjectures) on the strategies of the other agents. The agents then keep updating their conjectures with game theoretic and Bayesian statistical decision theoretic reasoning until a convergence of conjectures is achieved.\(^3\) In BEIC, the convergent conjectures are consistent with the equilibrium they support. BEIC therefore rules out equilibriums that are based on conjectures that are inconsistent with the equilibriums they support, as well as equilibriums supported by convergent conjectures that do not start with first order uninformative conjectures.

This difference in solution approach is related to an ongoing argument in games theory about the relative validity
of the two conflicting notions of rationality; Bayesian statistical decision theoretic rationality, and strategic rationality (as embodied by the game theoretic concepts of Nash equilibrium and its many refinements). Currently, games theorists think that the two concepts of rationality are incompatible. While most games theorists are steadfast to the concept of strategic rationality, this book undertakes the task of reconstructing the whole basic framework of non-cooperative games using the notion of Bayesian rationality. The specification of the process of conjecture formation in BEIC strengthens the concept of rationality used in games. The consequence is a new kind of game theoretic rationality that is based upon Bayesian rationality. This new game theoretic rationality includes rationality in actions and strategies, rationality in prior and posterior beliefs and, rationality in statistical decision rule.

The resulting Bayesian strand of games theory has a statistical decision theoretic foundation. It analyzes a larger set of games than the existing Nash equilibrium-based games theory given the same game theoretic structure. It also acts as an equilibrium selection criterion for the subset of games that the existing Nash equilibrium-based games theory analyzes. BEIC normally decreases the number of equilibriums to one, and selects the equilibrium that is most compelling. The BEIC approach therefore increases the analytical power of games theory in current applications. It also allows games theory to be applied to new areas, such as games of multi-sided incomplete information.

To comprehend the need for a new theory of non-cooperative games with Bayesian statistical decision theory as its foundation, one has to go back to the history of non-cooperative games theory and Bayesian statistical decision theory. Non-cooperative games theory started with the works of John Nash in the 1950s. Bayesian statistics
resurgence started in the 1970s and 1980s. Consequently, non-cooperative games theory developed largely independently of Bayesian statistical decision theory and had not started with a firm statistical decision theoretic foundation.

The works of Harsanyi (1967, 1968a, 1968b) came after the contributions of Nash (1950, 1951). Harsanyi’s (1967, 1968a, 1968b) works allow games of incomplete information to be analyzed. When Harsanyi first proposed his transformation of games of incomplete information, Kadane and Larkey (1982a, 1982b) criticized that the Harsanyi Bayesian games were not really Bayesian as they involved just the use of Bayes rule, but without the use of subjective probability and Bayesian decision theory in the decision-making process of the players.

This book will use subjective probability and Bayesian statistical decision theory to reconstruct non-cooperative games theory. Subjective probability is fundamental to BEIC. In the BEIC approach, the process of iterative conjecturing starts with first order uninformative conjectures. These first order uninformative conjectures are of course subjective, as are subsequent higher order conjectures.

What is the rationale to start with first order uninformative conjectures? Other than the assumption that the agents have no idea about the strategies adopted by other agents and the equilibrium they are in at the beginning of the conjecturing process, there are two further compelling reasons for starting with uninformative conjectures. First is the motive to let the game solve itself and select its own equilibrium strategies and convergent conjectures. The equilibrium so achieved is therefore not imposed or affected by informative conjectures arbitrarily chosen, but by the underlying structure and elements of the game, including the payoffs of the agents and the information they have. Second is to ensure that all
pathways and information sets have equal probabilities of being reached at the initial round of reasoning. That is to say, the conjecturing process explores every pathway and information set (either on or off equilibrium).

Harsanyi and Selten (1988) and Harsanyi (1995) propose a tracing procedure to select the most compelling equilibrium among multiple Nash equilibriums. Their tracing procedure starts with first order uninformative conjectures on the multiple Nash equilibriums. The solution of simultaneous games by Bayesian equilibrium by iterative conjectures (BEIC) is very similar to the tracing procedure of Harsanyi and Selten (1988). However, the BEIC approach does not start its tracing with only Nash equilibriums. It starts with all possible strategies of the players. This is ensured through the enforced use of first order uninformative conjectures on all possible actions or strategies.

The Bayesian iterative conjecture algorithm approach is especially suited to analyze sequential games with noisy inaccurate observations. An inaccurate observation is an observation with a noise term. This is the typical data that one encounters when doing statistical analysis. It is a generalization of perfect and imperfect information. The general case in noisy observation incomplete information sequential games is where both prior belief on action and the likelihood function from sample data play a role in the formation of posterior belief on action. The two special limiting cases are, first when the prior belief plays no role at all (as when the variance of the noise term becomes zero), and second when the likelihood function plays no role at all (as when the variance of the noise term reaches its maximum and therefore observation contains no information value). In other words, a noisy observation incomplete information sequential game is a general framework in which the perfect information and incomplete information sequential game
and imperfect information and incomplete information sequential game are special limiting cases.

The Bayesian iterative conjecture approach solution of simultaneous games is closely related to the focal point or Schelling point concept: it is about conjectures on the other agents’ actions and strategies, and conjectures on these conjectures, and the convergence of such conjectures. A process of conjecture starts with first order uninformative conjectures, and they are constantly being updated with game theoretic reasoning, but with no updating from likelihood functions since there are no observations on the actions of the other players. When all the processes of conjecturing in a game converge onto the same set of invariant distribution functions, there is a unique equilibrium.

Chapter Two presents the BEIC solution of sequential games with incomplete information and noisy inaccurate observation. It also proves that the Bayesian iterative conjecture algorithm is a Bayes decision rule (that is, an undominated decision rule) in a game theoretic context. Chapter Three presents the BEIC approach to sequential games of perfect and imperfect information. Chapter Four presents the BEIC approach to simultaneous games. Chapter Five concludes the book.

Notes

1. Refer to Nash (1950, 1951).
3. Refer to Teng (2012a).
5. Refer to Teng (2012b).
Sequential games with incomplete information and noisy inaccurate observation

2.1. Introduction

In current modeling of incomplete information games, there is normally either perfect or imperfect information. That is to say, either the action of the first mover is accurately observed by the later movers, or it is not observed at all. For instance, in a signaling game the action of the first moving player whose type is unknown is accurately observed by the second moving player. After observing the action of the first mover, the second mover uses game theoretic reasoning and the Bayes theorem to update his prior belief about the type of the first mover. He then chooses his optimal strategy given his posterior belief about the type of the first mover. The equilibrium so obtained is termed perfect Bayesian equilibrium. On the other hand, in an incomplete and imperfect information sequential game or an incomplete information simultaneous game, the action of the player whose type is unknown is not observed by the other player at all. The other player chooses his strategy given his prior belief about the type of the player with an unknown type. The equilibrium so obtained is termed Bayesian Nash equilibrium.
The assumption that the action of the first mover is either accurately observed or not observed at all is too restrictive. Given this assumption, there is no statistical inference and decision involved concerning the action of the first mover whose type is unknown. This is despite the use of Bayes theorem in updating beliefs about the possible types of the first moving player.

Sequential games with incomplete information and noisy inaccurate observation generalize the current sequential games framework in which there is either perfect information or imperfect information. Here the other player observes inaccurately the action of the player with an unknown type. Inaccurate observation means that the other player observes the action of the player with an unknown type with a noise term and there is a positive probability that he will make an observational error due to the noise term.

Since 1995, there have been many studies of games with noisy inaccurate observation. One of the focuses is the value of commitment by the first mover. There are also efforts to analyze games with incomplete information and noisy inaccurate observation. So far, however, no one has investigated the optimality of the statistical decision rules used in these games. This chapter will fill in this gap.

This chapter uses Bayesian equilibrium by iterative conjectures to analyze an inflation expectation game of incomplete information and noisy inaccurate observation. The Bayesian iterative conjecture algorithm allows beliefs to be endogenized in game theoretic context. This chapter also illustrates and proves that the Bayesian iterative conjecture algorithm decision rule is a Bayes (undominated) decision rule in game theoretic context, that is, the Bayesian iterative conjecture algorithm attains the supremum of the expected objective function (or the infimum of the expected loss function) of the agent making the inference.
In a sequential game with incomplete information and noisy inaccurate observations, the second mover must make statistical inferences on the actions of the first moving player with an unknown type. He does so based upon two sources of information. One source of information is the inaccurate observations on the actions of the first mover. This is the sample data. The other source of information is the evidence which concerns the motive of the first mover constructed through game theoretic reasoning, based upon knowledge such as the prior distribution function on the type of the first mover and the structure of the game. This source of information gives a belief about the probability of possible actions taken by the first mover. This belief is the prior predictive distribution function or conjecture on the actions of the first mover.

Given the need for statistical inferences and decisions, the player has to decide which statistical decision rule to use. Since in games theory the basic assumption is that the player is rational, that is, he optimizes, the decision rule has to be a Bayes decision rule. A decision rule is a Bayes decision rule if it attains the infimum of the expected loss function or the supremum of the expected utility function. Furthermore, given the knowledge a player has about the game, he will form a prior predictive distribution function on the possible actions of the other player. There are many ways to construct a prior distribution function. Therefore, in an incomplete information game with noisy inaccurate observations, there could be a large number of equilibriums given that there are many statistical decision rules and many different prior beliefs. The Bayesian iterative conjecture algorithm narrows down the number of equilibriums in such games normally to one through the use of first order uninformative conjectures.
2.2. An inflationary game

This section presents an inflationary expectation game with noisy inaccurate observations. There are two players: the government and the representative economic agent. The government moves first by setting the monetary growth rate, which determines the rate of inflation in the economy. The economic agent observes inaccurately the inflation rate due to a confounding noise term. Then the economic agent infers about the inflation rate using the Bayesian iterative conjecture algorithm. The structure of the game is common knowledge. Nature chooses the type of government from a predetermined prior distribution function which is common knowledge. Once chosen, the type of government is private knowledge, revealed to the government but not to the agent. The type of economic agent is common knowledge. Therefore, the economic agent has to make inferences on both the type and actions of the government. The distribution function of the noise term that confounds the observation of the economic agent of the actual inflation rate is common knowledge.

\( \pi \), the rate of inflation, is the action of the government. \( \Pi \), the inferred rate of inflation, is the action of the economic agent.

The payoff function of the economic agent is:

\[
V = - (\pi - \Pi)^3
\]

1

The payoff function of the agent is common knowledge.

The utility function of government is:

\[
U = a (\pi - \Pi) - \frac{\pi^2}{2}
\]

2

\( a \) measures the preference of the government for the employment of the stimulating effect of inflationary surprises. \( a \) decides the type of government. The government knows
the value of $a$ but the economic agent does not know the value of $a$. $a$ has a normal distribution which is common knowledge:

$$a \sim N(a, \gamma)$$

3

$a$ determines the willingness of the government to accept a tradeoff between higher inflation and a higher employment level. $a > 0$ means that the government has a preference for an inflation-generated increase in employment and output. $a < 0$ means that the government has a preference for deflation.

The action of the government is inaccurately observed by the economic agent with a noise term:

$$X = \pi + e$$

4

$X$ is the observation of the economic agent and $e$ is the noise term. $e$ has a normal distribution which is common knowledge:

$$e \sim N(0, \tau)$$

5

$e$ could be the change in real relative prices that confounds the observation on the inflation rate. The above leads to the following sampling distribution on $X$:

$$X|\pi \sim N(\pi, \tau)$$

6

Using the Bayesian iterative conjecture algorithm, the economic agent starts with a first order uninformative prior on the distribution of $\pi$. With an uninformative prior, Bayesian approach yields results that are the same as that of the maximum likelihood approach. So, with the first order uninformative prior, the economic agent solves:

$$\min_{\Pi} E(V) = \int_{-\infty}^{\infty} (\pi - \Pi)^2 f(\pi|X) d\pi$$

7
where $f(\pi | X)$ is the likelihood function:

$$\pi|X \sim N(X, \tau)$$

The first order condition is:

$$\frac{\partial E(V)}{\partial \Pi} = 2 \int (\pi - \Pi)f(\pi|X) d\pi = 0$$

The optimal solution is:

$$\Pi = X$$

The stochastic reaction function is:

$$\Pi|\pi \sim N(\pi, \tau).$$

The government, being the first mover in the game, anticipates the inference and reaction of the economic agent and solves:

$$\max_{\pi} E(U) = \int \left( a(\pi - X) - \frac{\pi^2}{2} \right) f(e) de = -\frac{\pi^2}{2}$$

The optimal solution is:

$$\pi = 0$$

The second order prior held by the agent is:

$$\pi = 0$$

This is a constant. Such being the case, the economic agent sets:

$$\Pi = 0$$

Given $\Pi = 0$, the government solves:

$$\max_{\pi} E(U) = \int \left( a(\pi - X) - \frac{\pi^2}{2} \right) f(e) de = a\pi - \frac{\pi^2}{2}$$
The optimal action for the government is:

\[ \pi = a \]  

The third order prior held by the economic agents is:

\[ \pi - N\left( \alpha, \gamma \right) \]  

Combining the prior and likelihood functions leads to the posterior distribution function of the economic agent on \( \pi \):

\[ \pi|X - N\left( \hat{\pi}, \psi \right) \]  

where

\[ \hat{\pi} = \frac{\gamma}{\gamma + \tau} X + \frac{\tau}{\gamma + \tau} \alpha a + (1 - \alpha) \hat{a} \]  

and

\[ \alpha = \frac{\gamma}{\gamma + \tau} \]  

and

\[ \hat{\psi} = \frac{\gamma \tau}{\gamma + \tau} \]  

In determining the optimal response to inflation, the economic agent solves:

\[ \min_{\Pi} E(V) = \int (\pi - \Pi)^2 f(\pi|X)d\pi \]  

The first order condition is:

\[ \frac{\partial E(V)}{\partial \Pi} = 2 \int (\pi - \Pi)^2 f(\pi|X)d\pi = 0 \]  

The optimal solution is:

\[ \Pi = \hat{\pi} = \alpha (\pi + e) + (1 - \alpha) \hat{a} \]
and the stochastic reaction function is:

$$\Pi | \pi - N\left(\alpha (\pi + e) + (1 - \alpha) a, \alpha^2 \tau\right)$$

26

The government, being the first mover in this inflationary belief game, anticipates the reaction of the economic agent. In determining the optimal inflation rate, the government solves:

$$\max_{\pi} E(U) = \int \left( a(\pi - X) - \frac{\pi^2}{2}\right) f(e) de$$

$$= a(1 - \alpha)\left(\pi - a\right) - \frac{\pi^2}{2}$$

27

The optimal solution is:

$$\pi = a(1 - \alpha)$$

28

The fourth order prior density function of the economic agent on the inflation rate is:

$$\pi \sim N\left(\bar{\pi}, \psi\right)$$

29

with mean

$$\bar{\pi} = a(1 - \alpha)$$

30

and variance

$$\psi = (1 - \alpha)^2 \gamma$$

31

Combining the prior and likelihood functions leads to the following posterior distribution function of the economic agent on $\pi$:

$$\pi | X \sim N\left(\bar{\pi}, \psi\right)$$

32
where
\[ \hat{\pi} = \frac{\psi}{\psi + \tau} X + \frac{\tau}{\psi + \tau} \pi = \alpha X + (1 - \alpha) \pi \]  
and
\[ \alpha = \frac{\psi}{\psi + \tau} \]  
and
\[ \hat{\psi} = \frac{\psi \tau}{\psi + \tau} \]  

In determining the optimal response to inflation, the economic agent solves:
\[ \min_{\Pi} E(V) = \int_{-\infty}^{\infty} (\pi - \Pi)^2 f(\pi|X) d\pi \]  
The first order condition is:
\[ \frac{\partial E(V)}{\partial \Pi} = 2 \int_{-\infty}^{\infty} (\pi - \Pi) f(\pi|X) d\pi = 0 \]  
The optimal solution is:
\[ \Pi = \hat{\pi} = \alpha(\pi + e) + (1 - \alpha) \hat{a} \]  
and the stochastic reaction function is:
\[ \Pi|\pi - N\left(\alpha(\pi + e) + (1 - \alpha) \hat{a}, \alpha^2 \tau\right) \]  
The government, being the first mover in this inflationary belief game, anticipates the reaction of the economic agent. In determining the optimal inflation rate, the government solves:
\[ \max_{\pi} E(U) = \int_{-\infty}^{\infty} \left(a(\pi - X) - \frac{\pi^2}{2}\right) f(e) de \]
\[ = a(1 - \alpha)\left(\pi - \hat{a}\right) - \frac{\pi^2}{2} \]
The optimal solution is:
\[ \pi = a(1 - \alpha) \]

The fifth order prior is therefore:
\[ \pi \sim N(\bar{\pi}, \psi) \]

with mean
\[ \bar{\pi} = a(1 - \alpha) \]

and variance
\[ \psi = (1 - \alpha)^2 \gamma \]

Note that the fifth order prior is identical to the fourth order prior. The conjectures converge at this point.

Now let the process of iterative conjecturing start from the other first order uninformative conjecture, the one which is held by the government about the action of the economic agent:

\[ \Pi \sim N(\Pi', \infty) \]

In the above equation, \( \Pi' \) is a constant and
\[ E(\Pi) = \Pi' \]

Given the first order uninformative conjecture, the government solves:

\[
\max \pi \ E(U) = \int_{-\infty}^{\infty} \left[ a(\pi - \Pi') - \frac{\pi^2}{2} \right] f(e) \, de \\
= a\pi - a\Pi' - \frac{\pi^2}{2}
\]

Since \( \Pi' \) is a constant, the optimal action for the government is:
\[ \pi = a \]
From here on, this process of conjectures is identical to the previous process of conjectures starting with the third order conjectures onward, and the two processes have the same convergent conjectures and equilibrium. This is not surprising given that in a noisy observation incomplete information game, the focus of conjecturing is on the prior distribution function of the action of the first mover.

At Bayesian equilibrium by iterative conjectures, the two equations that simultaneously determine $\psi$ and $\alpha$ are:

$$\psi = (1 - \alpha)^2 \gamma$$

and

$$\alpha = \frac{\psi}{\psi + \tau}$$

For analytical convenience, they are rewritten as:

G: $$\psi - (1 - \alpha)^2 \gamma = 0$$

H: $$\alpha(\psi + \tau) - \psi = 0$$

Using the implicit function theorem, the following results are derived:

$$\frac{\partial G}{\partial \psi} = 1 > 0$$

$$\frac{\partial G}{\partial \alpha} = 2(1 - \alpha)\gamma > 0$$

$$\frac{\partial H}{\partial \psi} = \alpha - 1 < 0$$

$$\frac{\partial H}{\partial \alpha} = \psi + \tau > 0$$
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\[ |J| = \begin{vmatrix} \frac{\partial G}{\partial \psi} & \frac{\partial G}{\partial \alpha} \\ \frac{\partial H}{\partial \psi} & \frac{\partial H}{\partial \alpha} \end{vmatrix} = \frac{\partial G}{\partial \psi} \frac{\partial H}{\partial \alpha} - \frac{\partial G}{\partial \alpha} \frac{\partial H}{\partial \psi} \]

\[ = (\psi + \tau) + 2(1 - \alpha)^2 \gamma > 0 \]

\[ \frac{\partial G}{\partial \gamma} = -(1 - \alpha)^2 < 0 \]

\[ \frac{\partial H}{\partial \gamma} = 0 \]

\[ \frac{\partial G}{\partial \tau} = 0 \]

\[ \frac{\partial H}{\partial \tau} = \alpha > 0 \]

\[ \frac{\partial \psi}{\partial \gamma} = \frac{1}{|J|} \left[ -\frac{\partial G}{\partial \gamma} \frac{\partial G/\partial \alpha}{\partial H/\partial \gamma} \right] = \frac{1}{|J|} (1 - \alpha)^2 (\psi + \tau) > 0 \]

\[ \frac{\partial \alpha}{\partial \gamma} = \frac{1}{|J|} \left[ \frac{\partial G}{\partial \psi} \frac{\partial G/\partial \gamma}{\partial H/\partial \psi} - \frac{\partial G/\partial \gamma}{\partial H/\partial \psi} \right] = \frac{-1}{|J|} (1 - \alpha)^3 > 0 \]

Greater uncertainty about the type of the government increases the prior uncertainty about the action of the government. This leads to a greater reliance on the data and lesser reliance on the prior.

\[ \frac{\partial \psi}{\partial \tau} = \frac{1}{|J|} \left[ -\frac{\partial G}{\partial \tau} \frac{\partial G/\partial \alpha}{\partial H/\partial \tau} \right] = \frac{1}{|J|} 2\alpha(1 - \alpha) \gamma > 0 \]

\[ \frac{\partial \alpha}{\partial \tau} = \frac{1}{|J|} \left[ \frac{\partial G}{\partial \psi} \frac{\partial G/\partial \gamma}{\partial H/\partial \psi} - \frac{\partial G/\partial \gamma}{\partial H/\partial \psi} \right] = \frac{-1}{|J|} \alpha < 0 \]
Greater variations in the noise term that clouds the observation lead to a lesser reliance on the data and a greater reliance on the prior. That leads to a greater variation in the inflation rate given the variation in the type of government, as the government takes advantage of the greater inaccuracy of inflationary observations by the economic agent. The economic agent anticipates this and adjusts his conjecture accordingly.

Figure 2.1 illustrates the relationship between the relative sizes of variance of the convergent prior distribution function on action and the variance of the distribution function of the noise term and the equilibrium of the game.

The forty-degree straight line through the origin is the reaction function of the agent when he bases his inference totally on the inaccurate observation and not at all on his prior distribution function on the action of the government. In this case: $\alpha = \frac{\Psi}{\Psi + \tau} = 1$. The line is at a tangent to the
in difference curve of the government at the point of origin. The equilibrium is \( \pi = \Pi = 0 \).

The horizontal straight line at \( \Pi = a \) is the reaction function of the agent when he bases his inference totally on the prior distribution function on the action of the government and not at all on his observations. In this case: \( \alpha = \frac{\psi}{\psi + \tau} = 0 \). The line is at a tangent to the indifference curve of the government at the point \( (\pi = a, \Pi = a) \). The equilibrium is \( \pi = \Pi = a \).

The straight line \( \alpha = \frac{1}{2} \) is the reaction function of the agent when he bases his inference equally on his inaccurate observations and the prior distribution function on the actions of the government. In this case: \( \alpha = \frac{\psi}{\psi + \tau} = \frac{1}{2} \). The line is at a tangent to the indifference curve of the government at the point \( (\pi = \frac{a}{2}, \Pi = \frac{a}{2}) \). The equilibrium is \( \pi = \Pi = \frac{a}{2} \).

**Indeterminacy of complete and perfect information sequential games**

This sub-section takes a new look at sequential games of complete and perfect information. It shows that the current understanding of sequential games of complete and perfect information as embodied in the solution method of backward induction is incomplete. It is just one of many possibilities.

To start with, what does it mean to have complete and perfect information? It means that the variance of the prior on type and the variance of the prior on action are both zero or are approaching zero. But then that raises the statistical decision theoretic question; which variance is smaller? Or to put it another way, which piece of information should the agent have greater confidence in; the prior on type or the
prior on action? The inflationary expectation game example presented previously provides an illustration.

By letting the noise term variance ($\tau$) tend to zero and the type prior distribution function variance ($\gamma$) tend to zero, the variance of the convergent prior distribution function on action ($\psi$) tends to zero as well. The noisy observation becomes a complete information game with perfect information. The equilibrium $\pi$ and $\Pi$ depend upon the value of $\lim_{\tau \to 0, \gamma \to 0} \alpha$.

If $\lim_{\tau \to 0, \gamma \to 0} \alpha = 1$, then $\Pi = \pi = 0$. This is the current backward induction solution of a sequential game of complete and perfect information. Note that in a sequential game of complete and perfect information, the agent solves:

$$\min_{\Pi} E(V) = (\pi - \Pi)^2$$  \hspace{1cm} 66

and sets

$$\Pi = \pi$$  \hspace{1cm} 67

The government solves:

$$\max_{\pi} U = a(\pi - \Pi) - \frac{\pi^2}{2} = a(\pi - \pi) - \frac{\pi^2}{2} = -\frac{\pi^2}{2}$$  \hspace{1cm} 68

and sets

$$\pi = 0$$  \hspace{1cm} 69

The solution is therefore $\Pi = \pi = 0$.

If $\lim_{\tau \to 0, \gamma \to 0} \alpha = 0$, then $\Pi = \pi = a$. This is the current solution of a sequential game of complete and imperfect information or a simultaneous game. Note that in a sequential game of complete and imperfect information (and simultaneous game as well), the agent solves:

$$\min_{\Pi} E(V) = (\pi - \Pi)^2$$  \hspace{1cm} 70
and sets

$$\Pi = \pi$$

The government solves:

$$\max_{\pi} U = a(\pi - \Pi) - \frac{\pi^2}{2}$$

and sets

$$\pi = a$$

The solution is therefore $$\Pi = \pi = a$$.

However, $$\lim_{\tau, \gamma \to 0} \alpha$$ could take on any value from 0 to 1. For instance, it could be that $$\lim_{\tau, \gamma \to 0} \alpha = 0.5$$. In this case, $$\Pi = \pi = \frac{a}{2}$$. Figure 2.1 illustrates the three cases of $$\lim_{\tau, \gamma \to 0} \alpha$$ equal to 0, 0.5 and 1.

The relative sizes of the variance of the convergent equilibrium prior distribution function on the action of player 1 and the variance of the distribution function of the noise term decide the value of $$\alpha$$. It therefore also decides whether the equilibrium of the noisy inaccurate observation sequential game will be closer to the perfect information game or the imperfect information game. This is true even when both variances approach zero in absolute size.

The example illustrates that the incomplete information and noisy inaccurate observation sequential game generalizes the two distinct frameworks of incomplete information sequential games with imperfect information or incomplete information simultaneous games (with Bayesian Nash equilibrium as the dominant solution concept), and incomplete information sequential games with perfect information (with perfect Bayesian equilibrium as the dominant solution concept).

The value of $$\lim_{\tau, \psi \to 0} \alpha$$ could be any point in $$[0,1]$$. This gives rise to indeterminacy in the equilibrium of the game.
The root cause is that the relative weight to be assigned to conjectures or observations in the Bayesian statistical decision process of this game has not been specified. Viewing complete and perfect information as a limiting case of incomplete information and noisy inaccurate observation enables one to discern the issue of statistical decision theoretic indeterminacy in games of complete and perfect information.

**Efficiency of decision rule**

This sub-section gives an illustration of the efficiency of the Bayesian iterative conjecture algorithm as a decision rule. It compares the expected payoff of an agent using the Bayesian iterative conjecture algorithm to that of an agent using the maximum likelihood approach, under the assumption that the decision rule of the agent making the inference is common knowledge. It shows that the agent using the Bayesian iterative conjecture approach attains a higher expected payoff than the agent using the maximum likelihood approach.

The expected utility of the economic agent in the Bayesian equilibrium by iterative conjectures is:

$$ EE \left( V_p \right) = - EE \left( \frac{\left( a(1-\alpha) - \alpha (a(1-\alpha) + e)^2 \right)}{\left( 1 - \alpha \right)^2 a} \right)^2 $$

$$ = -\left( 1 - \alpha \right)^4 \gamma - \alpha^2 \tau $$

If instead of the Bayesian iterative conjecture algorithm, the second mover makes inferences and decisions using the maximum likelihood estimation approach, and this is
common knowledge, what is his expected utility? The economic agent makes use of the likelihood function:

\[ \pi | X - N(X, \tau) \]

and sets

\[ \Pi = X \]

The government, being the first mover in the game, anticipates the inference and reaction of the economic agent and solves:

\[ \max_\pi E(U) = \int a(\pi - X) - \frac{\pi^2}{2} \cdot f(e) \, de = -\frac{\pi^2}{2} \]

The optimal solution is:

\[ \pi = 0 \]

The expected payoff of the economic agent is:

\[ EE(V_{mle}) = -EE(0 - (0 + e))^2 = -\tau \]

Note that

\[ EE(V_{mp}) - EE(V_{mle}) = -(1 - \alpha)^3 \gamma - \alpha^2 \tau + \tau \]

\[ = -(1 - \alpha)\left[(1 - \alpha)^3 \gamma - (1 + \alpha) \tau\right] = -(1 - \alpha) \tau > 0 \]

since

\[ 1 - \alpha = \frac{\tau}{\psi + \tau} = \frac{\tau}{(1 - \alpha)^3 \gamma + \tau} \]

and therefore

\[ (1 - \alpha)^3 \gamma - \alpha \tau = 0 \]

The economic agent who uses the Bayesian iterative conjecture decision rule attains a higher payoff than the economic agent who uses the maximum likelihood approach.
2.3. Bayesian iterative conjecture algorithm as a Bayes decision rule

This section formally proves that the Bayesian equilibrium iterative conjecture approach is a Bayes (undominated) decision rule for sequential games with incomplete information and noisy inaccurate observations under the assumption that the decision rule of the player making the inference is common knowledge. A statistical decision rule is a Bayes decision rule if it attains the infimum of the expected loss function or the supremum of the expected utility function and therefore there is no other decision rule that attains a lower expected loss or higher expected utility.

**Proposition 1:**
*The Bayesian iterative conjecture algorithm is a Bayes (undominated) statistical decision rule when the statistical decision rule of the player making the inference is common knowledge.*

**Proof:**
Let $A_1$ be the action space of the first mover, $A_2$ be the action space of the second mover, $\varepsilon$ be the noise term that confounds the first mover’s action when observed by the second mover, $T_1$ be the type space of the first mover, $u_1$ be the payoff function of the first mover, $u_2$ be the payoff function of the second mover, and $f(\varepsilon)$ be the distribution function of the noise term, which is common knowledge though the realization of the noise term is known by none. The type of the first mover $t_1$ is privately known by the first mover. The second mover’s belief on the type of the first mover $p_2(t_1)$ describes his uncertainty about the first mover’s possible type. The likelihood function $l_2(a_1 + \varepsilon | a_1)$ describes the second mover’s
uncertainty about the action of the first mover given the noisy observation on the action of the first mover.

In Bayesian equilibrium by iterative conjectures, the convergent prior distribution function best predicts the strategy of the first mover given his type. That is,

$$f_z (a_i) = a^*_1 (t_1) p_2 (t_1)$$

where $a^*_1 (t_1)$ is the optimal course of action for the first mover given his type. The second mover updates his prior belief using the Bayes’ rule. That is:

$$\pi_2 (a_1 | a_1 + \varepsilon) = \frac{f_z (a_1) l_z (a_1 + \varepsilon | a_1)}{\sum_{a_1 \in A_1} f_z (a_1) l_z (a_1 + \varepsilon | a_1)}$$

if the action space is discrete and:

$$\pi_2 (a_1 | a_1 + \varepsilon) = \frac{f_z (a_1) l_z (a_1 + \varepsilon | a_1)}{\int_{a_1 \in A_1} f_z (a_1) l_z (a_1 + \varepsilon | a_1)}$$

if the action space is continuous.

Since the first mover knows the decision rule of the second mover, in Bayesian equilibrium by iterative conjectures, $a^*_1 (t_1)$ solves:

$$\max_{a_1 \in A_1, \varepsilon} \sum_{t_1} u_1 (a_1, a^*_1 (a_1, \varepsilon); t_1)$$

if the action space and noise space are discrete or:

$$\max_{a_1 \in A_1, \varepsilon} \int_{\varepsilon} u_1 (a_1, a^*_2 (a_1, \varepsilon); t_1) f (\varepsilon) d\varepsilon$$

if the action space and noise space are continuous.

In Bayesian equilibrium by iterative conjectures, $a^*_2 (a_1, \varepsilon)$ solves:

$$\max_{a_1 \in A_1, \varepsilon} \sum_{a_2 \in A_2} u_2 (a_2, a^*_1 (t_1)) \pi_2 (a^*_1 (t_1) | a^*_1 + \varepsilon)$$
where
\[
\pi_2\left(\bar{a}_2(t_1), a_1^* \mid a_1^* + \epsilon\right) = \frac{f_2\left(\bar{a}_2(t_1), a_1^* \mid a_1(t_1) + \epsilon\right)l_2\left(a_1^* \mid a_1(t_1) + \epsilon\right)}{\sum_{a_1 \in A_1} f_2\left(\bar{a}_2(t_1), a_1 \mid a_1(t_1) + \epsilon\right)l_2\left(a_1 \mid a_1(t_1) + \epsilon\right)}
\]

if the action space is discrete or:
\[
\max_{a_2 \in A_2} \int_{a_1 \in A_1} u_2(a_2, a_1^* \mid t_1) \pi_2\left(\bar{a}_2(t_1), a_1^* \mid a_1^* + \epsilon\right)da_1
\]

where
\[
\pi_2\left(\bar{a}_2(t_1), a_1^* \mid a_1^* + \epsilon\right) = \frac{f_2\left(\bar{a}_2(t_1), a_1^* \mid a_1(t_1) + \epsilon\right)l_2\left(a_1^* \mid a_1(t_1) + \epsilon\right)}{\int_{a_1 \in A_1} f_2\left(\bar{a}_2(t_1), a_1 \mid a_1(t_1) + \epsilon\right)l_2\left(a_1 \mid a_1(t_1) + \epsilon\right)da_1}
\]

if the action space is continuous.

Therefore, the expected utility of the second mover using the Bayesian equilibrium by iterative conjectures decision rule is:
\[
\sum_{a_1 \in A_1} u_2(a_2, a_1^* \mid t_1) \pi_2\left(\bar{a}_2(t_1), a_1^* \mid a_1^* + \epsilon\right)
\]

\[
= \sup_{a_1 \in A_1} \sum_{a_1 \in A_1} u_2(a_2, a_1^* \mid t_1) \pi_2\left(\bar{a}_2(t_1), a_1^* \mid a_1^* + \epsilon\right)
\]

if the action space is discrete or:
\[
\int_{a_1 \in A_1} u_2(a_2, a_1^* \mid t_1) \pi_2\left(\bar{a}_2(t_1), a_1^* \mid a_1^* + \epsilon\right)da_1
\]

\[
= \sup_{a_1 \in A_1} \int_{a_1 \in A_1} u_2(a_2, a_1^* \mid t_1) \pi_2\left(\bar{a}_2(t_1), a_1^* \mid a_1^* + \epsilon\right)da_1
\]

if the action space is continuous.

Q.E.D.

Proposition 1 ensures the player who needs to make statistical inferences and decisions has no incentive to choose other approaches over Bayesian equilibrium by the iterative conjecture approach.
2.4. Conclusions

Bayesian equilibrium by iterative conjectures achieves rationality not only in action/strategy and beliefs, both prior and posterior. It also attains rationality in (statistical) decision rule. It narrows down the number of equilibriums, normally to one, by eliminating equilibriums that are inconsistent with their associated beliefs, as well as equilibrium consistent beliefs that do not start with first order uninformative conjectures. Consequently, as shown in the inflationary expectation game example, it allows insightful and interesting comparative static exercises to be performed for both beliefs and strategies.

Bayesian equilibrium by iterative conjectures approach is a Bayes decision rule, that is, it is not dominated by any other decision rule. Therefore, players making statistical inferences and decisions have no incentive to prefer other decision rules to them. That is to say, Bayesian equilibrium by iterative conjectures is firmly grounded on the foundation of Bayesian statistical decision theory. That firm statistical decision theoretic foundation also allows Bayesian equilibrium by the iterative conjecture approach to discern the indeterminacy of a complete and perfect information game hitherto not noticed by games theory researchers and practitioners.

Notes

2. Refer to Maggi (1999) and de Huan, Offerman and Sloof (2011).
3. Other criteria for selecting decision rule include the minimax rule and admissibility. Refer to Berger (1980).
Sequential games with perfect and imperfect information

3.1. Introduction

This chapter uses the Bayesian iterative conjecture algorithm to solve sequential games with perfect information, and sequential games with imperfect information. It also discusses the relationships between Bayesian equilibrium by iterative conjectures and sub-game perfect equilibrium and perfect Bayesian equilibrium. The focus of the chapter is on the solution of sequential games with incomplete and perfect information by the Bayesian iterative conjecture approach. The chapter illustrates the power of the Bayesian iterative conjecture algorithm by solving a two-sided incomplete and perfect information sequential game, a daunting task to the current perfect Bayesian equilibrium-based approach.

In the current Nash equilibrium-based games theory, the solution algorithm of sequential games with incomplete and perfect information solves a game by checking if a combination of strategies and (posterior) beliefs (on types) of players constitutes an equilibrium. Implicit in this algorithm is the assumption that the players know which equilibrium they are in and know the equilibrium strategies and beliefs of the other players. The assumption that players know the equilibrium of the game and strategies and beliefs
of the other players removes much of the inherent uncertainty about the strategies of the other players in games of incomplete information. This makes computation easy and gives the method its popular appeal. However, the assumption that the players know the equilibrium that they are in does not make sense when there are multiple equilibriums. And yet, by the perfect Bayesian equilibrium approach, sequential games with incomplete and perfect information typically have multiple equilibriums.

There is another problem with the current Nash equilibrium-based approach in the solution of sequential games with incomplete and perfect information. In a sequential game of incomplete and perfect information, there is uncertainty about the type of some of the players. Therefore, despite the fact that the player observes the actions of a player of an unknown type perfectly, he must still infer about the strategy of each type through game theoretic reasoning. Also, the player of an unknown type must also conjecture about the strategy and conjectures of the other players when selecting his strategy. Consequently, unlike a sequential game of complete and perfect information, the player cannot condition his strategy upon the other players’ strategy: the player of a known type cannot do so as the other players have more than one type, and the player with a known type observes the other players’ actions but not their strategy, and the player of an unknown type cannot do so as he has to infer about the conjectures or beliefs (which he does not observe) and strategies (which depends upon the conjectures) of the other players.

The Bayesian iterative conjecture algorithm approach, in contrast, investigates and models how the conjectures of players about the strategies of the other players and their conjectures converge. The solution algorithm for sequential
Sequential games with perfect and imperfect information

Games with incomplete and perfect information is exactly the same as that for the previous chapter's incomplete information and noisy inaccurate observation sequential games. However, for sequential games with incomplete but perfect information, since the players make perfect observations of the action (perfect information), there is no need to make a Bayesian statistical inference on it. Although, there is still the need to make Bayesian statistical inferences and decisions on the strategies and types of other players.

In solving sequential games with incomplete and perfect information, the Bayesian iterative conjecture algorithm approach starts with first order uninformative conjectures, assuming that players do not know the other players’ strategy or the equilibrium of the game, though they observe perfectly the actions of the other players. Conjectures are updated using game theoretic and Bayesian statistical decision theoretic reasoning until a convergence emerges which then defines the BEIC.

Another important difference between the BEIC approach and the perfect Bayesian equilibrium-based approach is that when having pooling equilibrium, current games theory needs to specify probability beliefs on off equilibrium paths. When Harsanyi (1967, 1968a, b) first introduced his transformation of incomplete information games and called them Bayesian games, Kadane and Larkey (1982a, b) raised the objection that these games are not really Bayesian as there is no subject probability. Harsanyi (1982a, b) replied that he was an objective Bayesian who believed that only an objective prior should be used. Yet, this issue soon resurfaced. Subject probability was resurrected in the Harsanyi incomplete information games in the form of off equilibrium beliefs.

In contrast to perfect Bayesian equilibrium and its many refinements, the BEIC approach uses a hierarchy of
conjectures; first order uninformative prior conjectures and higher order conjectures, the highest order conjectures being the set of convergent conjectures (if it exists). The enforced use of first order uninformative conjectures ensures that all possible pathways are considered and no pathway is left unexplored. Consequently, the convergent conjectures and their corresponding equilibrium, either separating or pooling equilibrium, are supported by lower level conjectures, and there is no need to specifically determine off equilibrium path beliefs as they are already given within the hierarchy of conjectures.

3.2. The Bayesian iterative conjecture algorithm, sub-game perfect equilibrium and perfect Bayesian equilibrium

Sequential games with perfect information are typically solved through backward induction. This section illustrates that the Bayesian iterative conjecture algorithm could handle the task equally well. This section shows that the Bayesian iterative conjecture algorithm eliminates equilibriums based upon non-credible threats, and achieves the objective of sub-game perfect equilibrium refinement. The section also shows that the Bayesian iterative conjecture algorithm is a stronger selection criterion than sub-game perfect equilibrium.

Consider the following example:
Sequential games with perfect and imperfect information

The two pure strategy Nash equilibriums are (D, L) and (U, R). Yet, (U, R) is based upon a non-credible threat.

The BEIC solution:

**Figure 3.1** A sequential game

**Table 3.1** Normal form of the game in Figure 3.1

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
<th>L (y)</th>
<th>R (1-y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U (x)</td>
<td>3, 5</td>
<td>3, 5</td>
<td></td>
</tr>
<tr>
<td>D (1-x)</td>
<td>5, 4</td>
<td>1, 2</td>
<td></td>
</tr>
</tbody>
</table>

The two pure strategy Nash equilibriums are (D, L) and (U, R). Yet, (U, R) is based upon a non-credible threat.

The BEIC solution:

**Table 3.2** BEIC for the game in Figure 3.1

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Probability X</th>
<th>Probability Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
The BEIC approach rules out the Nash equilibrium based on a non-credible threat. The Bayesian iterative conjecture algorithm achieves the objective of the refinement of sub-game perfect equilibrium.

Consider the following game:

There are two (pure strategy) Nash equilibriums, (U, R) and (D, L). Both are sub-game perfect equilibriums as there is no sub-game. However, R is a weakly dominated strategy. The BEIC approach eliminates (U, R) and is a stronger selection criterion than sub-game perfect equilibrium.

![Figure 3.2](image)

**Figure 3.2** An imperfect information game

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L (y)</td>
</tr>
<tr>
<td>U (x)</td>
<td>5, 5</td>
</tr>
<tr>
<td>D (1-x)</td>
<td>8, 1</td>
</tr>
</tbody>
</table>

**Table 3.3** Normal form of the game in Figure 3.2
Bayesian equilibrium by iterative conjectures is a refinement of perfect Bayesian equilibrium.

**Definition 1:**

In a sequential game of incomplete information with $N$ players, and each player with $N_k$ types, there are $T = \sum_{k=1}^{N} N_k$ possible types of players. As each type of each player has $T_l$ strategies, there are $S = \sum_{l=1}^{T} T_l$ sets of strategies, $A_i$, where $i = 1, 2, 3, ..., S$. The player $k$ chooses action $a_i \in A_i$. There are $T$ utility functions, $U_i(a_1, a_2, ..., a_S)$.

A set of priors or conjectures in this game specifies the probability distribution of each $a_i \in A_i$ by $p_i$, where $\sum p_i = 1$ if the strategy space is discrete and $\int p_i = 1$ if the strategy space is continuous.

The Bayesian equilibrium by iterative conjectures of such a game has $S$ first order uninformative priors or conjectures, denoted as $p_i^1$, where the superscript identifies the order of conjectures. An uninformative prior or conjecture in this game assigns to each $a_i \in A_i$ equal probability.

Given, $p_i^1$, if an $\tilde{A}_i$ exists and $\tilde{a}_i \in \tilde{A}_i \in A_i$ and

$$U_i(a_1, a_2, ..., \tilde{a}_i, ..., a_S; p_i^1, p_1^1, ..., p_S^1) > U_i(a_1, a_2, ..., a_i, ..., a_S; p_i^1, p_1^1, ..., p_S^1)$$

---

**Table 3.4** BEIC for the game in Figure 3.2

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Probability X</th>
<th>Probability Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

---
where \( a_i \notin \tilde{A}_i \), then \( p_i^2 \) is updated such that probability one is assigned to \( \tilde{A}_i \) and equal probability is assigned to individual \( \tilde{a}_i \in \tilde{A}_i \). In the case where the set \( \tilde{A}_i \) has only one element which is \( \tilde{a}_i \), and 

\[
U_i(a_1, a_2, ..., a_i, ..., a_5; p_1^1, p_2^1, ..., p_5^1) > U_i(a_1, a_2, ..., \tilde{a}_i, ..., a_5; p_1^1, p_2^1, ..., p_5^1),
\]

then \( p_i^2 \) is updated such that probability one is assigned to \( \tilde{a}_i \). In the case where the set of \( \tilde{A}_i \) equals the set of \( A_i \), then there is no updating and \( p_i^2 = p_i^1 \).

The updating of higher order conjectures from \( p_i^2 \) to \( p_i^3 \) and onwards, follows the same procedure. A BEIC is achieved when there is a convergence in the conjectures, that is, \( p_i = p_i^{r+1} = p_i^{r+2} = p_i^{r+3} = ... \forall i \).

Recall that in a perfect Bayesian equilibrium, players must have beliefs about the strategies of the other players, and the players’ strategies must be optimal given the players’ beliefs and the other players’ subsequent strategies; on the equilibrium path, beliefs are determined by the players’ equilibrium strategies and off the equilibrium path, beliefs are determined by the players’ equilibrium strategies where possible.\(^1\)

**Proposition 2:**

BEIC is a refinement of perfect Bayesian equilibrium.

**Proof:**

As the set of inequalities

\[
U_i\left(a_1, a_2, ..., \tilde{a}_i, ..., a_5; p_1, p_2, ..., p_5\right) > U_i\left(a_1, a_2, ..., \tilde{a}_i, ..., a_5; p_1, p_2, ..., p_5\right)
\]

holds for all \( i \) given the set of \( p_i \) which specifies the probability distribution of \( a_i \), and \( p_i = p_i^{r+1} = p_i^{r+2} = p_i^{r+3} = ... \forall i \) in a Bayesian equilibrium by iterative conjectures, the beliefs
are consistent with the equilibrium and the associated strategies. A Bayesian equilibrium by iterative conjectures is therefore a perfect Bayesian equilibrium.

Furthermore, the restriction that the conjectures starting with first order uninformative conjectures narrows down the set of equilibriums of BEIC to a sub-set of perfect Bayesian equilibrium. BEIC is therefore a refinement of perfect Bayesian equilibrium.

Q. E. D.

To illustrate the above point, consider the game in Figure 3.3:

![Figure 3.3](image)

**Figure 3.3** A sequential game with imperfect information $1^2$

**Table 3.5** Normal form representation of the game in Figure 3.3

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
<th>L' (z)</th>
<th>R' (1-z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L (x)</td>
<td></td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>M (y)</td>
<td></td>
<td>0, 2</td>
<td>0, 1</td>
</tr>
<tr>
<td>R (1-x-y)</td>
<td></td>
<td>1, 3</td>
<td>1, 3</td>
</tr>
</tbody>
</table>

The BEIC solution specifies the strategies and beliefs of the players at equilibrium.
As another illustration, consider the game in Figure 3.4.
Let the probability that player 1 plays D be \( x \) and the probability that he plays A be \( 1-x \), the probability that player 2 plays L be \( y \) and the probability that he plays R be \( 1-y \) and, the probability that player 3 plays L' be \( z \) and the probability that he plays R' be \( 1-z \).

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>( \text{Pr (x)} )</th>
<th>( \text{Pr (y)} )</th>
<th>( \text{Pr (z)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Figure 3.4** A sequential game with imperfect information 2\(^3\)
Table 3.7  Normal form representation of the game in Figure 3.4 when player 1 plays D

<table>
<thead>
<tr>
<th>Player 3</th>
<th>Player 2</th>
<th>L' (z)</th>
<th>R' (1-z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L (y)</td>
<td>1, 2, 1</td>
<td>3, 3, 3</td>
<td></td>
</tr>
<tr>
<td>R (1-y)</td>
<td>0, 1, 2</td>
<td>0, 1, 1</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.8  Normal form representation of the game in Figure 3.4 when player 1 plays A

<table>
<thead>
<tr>
<th>Player 3</th>
<th>Player 2</th>
<th>L' (z)</th>
<th>R' (1-z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L (y)</td>
<td>2, 0, 0</td>
<td>2, 0, 0</td>
<td></td>
</tr>
<tr>
<td>R (1-y)</td>
<td>2, 0, 0</td>
<td>2, 0, 0</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.9  BEIC solution of the game in Figure 3.4

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Pr (x)</th>
<th>Pr (y)</th>
<th>Pr (z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The BEIC solution specifies the strategies and beliefs of the players at equilibrium:

Consider the game in Figure 3.5. There are two pure strategy Nash equilibriums: (L1, R2, R3) and (R1, R2, L3). (R1, R2, L3) is a perfect Bayesian equilibrium and sequential equilibrium, while (L1, R2, R3) is not. (L1, R2, R3) prescribes a non-optimal strategy R2 for player 2.

Let the probability that player 1 plays L1 be x, the probability that player 2 plays L2 be y and the probability that player 3 plays L3 be z. The BEIC approach selects the perfect Bayesian equilibrium among Nash equilibriums.
3.3 Solving sequential games of incomplete and perfect information

This section compares the solutions achieved under the BEIC approach and the perfect Bayesian equilibrium approach (including its two popular refinements, sequential equilibrium and the intuitive criterion). The BEIC approach has at least three advantages over the perfect Bayesian equilibrium.
approach: it normally generates only an intuitively compelling unique equilibrium; it does not involve the ad hoc specification of off-equilibrium beliefs; and it resolves inconsistencies in equilibrium results between sub-game perfect equilibrium (by backward induction) and perfect Bayesian equilibrium (and sequential equilibrium and intuitive criterion).

Perfect Bayesian equilibrium solves a sequential game of incomplete information by asking if a certain combination of strategies and beliefs of the players constitute an equilibrium. The process of how the players form their beliefs is not specified, except the requirements that they be consistent with the equilibrium strategies of the players and abide by the Bayes’ theorem in updating. A set of strategies and beliefs constitute a perfect Bayesian equilibrium, no matter how the beliefs came into being. Consequently, typically there exist multiple equilibriums and many refinements are resorted to for narrowing down the set of equilibriums.

The BEIC approach, in contrast, solves a sequential game with incomplete and perfect information by starting from the assumption that players neither know the other players’ strategy nor the equilibrium of the game through the use of first order uninformative conjectures. An uninformative conjecture assigns equal probabilities to all possible courses of actions or strategies. Conjectures are then updated using game theoretic and Bayesian statistical decision theoretic reasoning (including but not confined to the use of Bayes’ theorem) until a convergence of conjectures emerges which then defines a BEIC. The BEIC approach therefore produces a hierarchy of conjectures: first order uninformative prior conjectures and higher order conjectures, with the highest order conjectures being the set of convergent conjectures (if it exists).

There are two compelling reasons to start with first order uninformative conjectures. One is to let the game solve itself
and selects its own equilibrium strategies and convergent conjectures, rather than having the equilibriums being dictated by the informative first order conjectures of the agents. The natural emergence of a set of convergent informative conjectures and its associated equilibrium strategies from the uninformative first order conjectures is the quintessence of the Bayesian equilibrium by the iterative conjecture approach.

The second reason for using the first order uninformative conjectures is to ensure that every information set and pathway is reached with equal probability for the initial rounds of conjectural reasoning, so that the process of Bayesian iterative conjecture reasoning explores every possible path. This inherent built-in mechanism of the BEIC approach contrasts sharply with the handling of off equilibrium path beliefs in the perfect Bayesian equilibrium approach, which either leaves the off equilibrium path beliefs undefined, or refines them by myriads of ad hoc refinement criteria. That is to say, the specification of off equilibrium path beliefs is a major problem for the perfect Bayesian equilibrium approach. In contrast, in Bayesian equilibrium by iterative conjectures, the convergent conjectures are supported by lower level conjectures, the lowest level conjectures being the uninformative first order conjectures, which cover all possible strategies and pathways, and therefore off equilibrium paths beliefs are automatically generated in the process and there is no need for ad hoc specification of off equilibrium beliefs.

Very often there are too many perfect Bayesian equilibriums. Consequently, games theorists have invented many refinements. Sequential equilibrium is probably the most popular refinement of perfect Bayesian equilibrium. Yet even sequential equilibrium is still at times too weak for equilibrium selection. Among the many refinements of sequential equilibrium, the
intuitive criterion stood out for its compelling reasoning and widespread applications, especially in signaling games. The following examples contrast the two approaches, the Bayesian equilibrium by iterative conjectures and the perfect Bayesian equilibrium (and sequential equilibrium) approach as refined by the intuitive criterion.

**Coordination game**

Consider the signaling game as depicted in Figure 3.6.

The probability of player 1 being type 1 and type 2 is $r$ and $1-r$. There are four perfect Bayesian equilibriums:

i. (L, R; u(L), d(R)). This equilibrium is socially suboptimal.

ii. (R, L; d(L), u(R)). This equilibrium is socially optimal.
iii. Pooling equilibrium (R, R; u(L), u(R); r>\frac{1}{6}, p>\frac{5}{6}).

iv. Pooling equilibrium (L, L; d(L), d(R); r<\frac{5}{6}, q<\frac{1}{6}).

The two pooling equilibriums are ruled out by the intuitive criterion. The separating equilibriums do not change as r changes.

Solving by the BEIC approach: let the probability that the receiver plays U when L is observed be a and the probability that the receiver plays U when R is observed be b. Let the probability that the type 1 sender plays L be x and the probability that the type 2 sender plays L be y.

The reaction functions are:

Type 1 sender plays L if \(2a + (1-a) > 5b\) or \(\frac{1}{5} + \frac{a}{5} > b\). Type 1 sender is indifferent between playing L or playing R if \(2a + (1-a) = 5b\) or \(\frac{1}{5} + \frac{a}{5} = b\). Type 1 sender plays R if \(2a + (1-a) < 5b\) or \(\frac{1}{5} + \frac{a}{5} < b\).

Type 2 sender plays L if \(5(1-a) = b + 2(1-b)\) or \(\frac{3}{5} + \frac{b}{5} > a\). Type 2 sender is indifferent between playing L or playing R if \(5(1-a) = b + 2(1-b)\) or \(\frac{3}{5} + \frac{b}{5} = a\). Type 2 sender plays R if \(5(1-a) < b + 2(1-b)\) or \(\frac{3}{5} + \frac{b}{5} < a\).

When L is observed, the receiver plays U if \(2xr > xr + 5y(1-r)\) or \(xr > 5y(1-r)\). The receiver is indifferent between playing U or D if \(2xr = xr + 5y(1-r)\) or \(xr = 5y(1-r)\). The receiver plays D if \(2xr < xr + 5y(1-r)\) or \(xr > 5y(1-r)\).

When R is observed, the receiver plays U if \(5(1-x) r + (1-y)(1-r) > 2(1-y)(1-r)\) or \(5(1-x) r > (1-y)(1-r)\). When R is observed, the receiver is indifferent between
playing U or playing D if \(5(1 - x) r + (1 - y)(1 - r) = 2(1 - y)(1 - r)\) or if \(5(1 - x) r = (1 - y)(1 - r)\). When R is observed, the receiver plays D if \(5(1 - x) r + (1 - y)(1 - r) < 2(1 - y)(1 - r)\) or if \(5(1 - x) r < (1 - y)(1 - r)\).

The Bayesian iterative conjecture approach solutions are:

When \(r < 1/6\), given the first order conjectures that \(x = 1/2\) and \(y = 1/2\), the receiver plays D when L is observed and D when R is observed. Anticipating that, type 1 sender plays L and type 2 sender plays L. The receiver updates his conjectures to \(x = 1\) and \(y = 1\) and plays D(L). Anticipating that, type 1 sender plays L and type 2 sender plays L. The conjectures converge here. The BEIC is \((L, L; D(L), D(R))\).

When \(r = 1/6\), given the first order conjectures that \(x = 1/2\) and \(y = 1/2\), the receiver plays D when L is observed and is indifferent between U and D when R is observed. Anticipating that, type 1 sender plays R and type 2 sender plays L. The receiver updates his conjectures to \(x = 0\) and \(y = 1\) and plays D(L) and U(R). Anticipating that, type 1 sender plays R and type 2 sender plays L. At this point the conjectures converge. The BEIC is \((R, L; D(L), U(R))\).

When \(1/6 < r < 5/6\), given the first order conjectures that \(x = 1/2\) and \(y = 1/2\), the receiver plays D when L is observed and U when R is observed. Anticipating that, type 1 sender plays R and type 2 sender plays L. The receiver updates his conjectures to \(x = 0\) and \(y = 1\) and plays D(L) and U(R). Anticipating that, type 1 sender plays R and type 2 sender plays L. The conjectures converge here. The BEIC is \((R, L; D(L), U(R))\).

When \(r = 5/6\), given the first order conjectures that \(x = 1/2\) and \(y = 1/2\), the receiver is indifferent between U and D when L is observed and plays U when R is observed. Anticipating that, type 1 sender plays R and type 2 sender plays L. The receiver updates his conjectures to \(x = 0\) and \(y = 1\) and plays D(L) and U(R). Anticipating that, type 1 sender plays R and
type 2 sender plays L. The conjectures converge at this point. The BEIC is \((R, L; D(L), U(R))\).

When \(r > 5/6\), given the first order conjectures that \(x = 1/2\) and \(y = 1/2\), the receiver plays \(U\) when \(L\) is observed and \(U\) when \(R\) is observed. Anticipating that, type 1 sender plays \(R\) and type 2 sender plays \(R\). The receiver updates his conjectures to \(x = 0\) and \(y = 0\) and plays \(U(R)\). The conjectures converge here. The BEIC is \((R, R; D(L), U(R))\).

By the BEIC approach, the selection of equilibriums depends on the value of \(r\). In contrast, the PBE approach has separating equilibriums that have nothing to do with \(r\). The two extreme cases of \(r\) approaching one and \(r\) approaching zero help to shed light on this distinction between the two approaches. When \(r\) approaches one, the BEIC is \((R, R; U(L), U(R))\). It is equilibrium iii of the PBE approach, which is ruled out by the intuitive criterion. Note that the BEIC for this limiting case agrees with the equilibrium for the sequential complete and perfect information game, which is represented by figure 3.7.

![Figure 3.7](image.png)

**Figure 3.7** The game in Figure 3.6 when \(r\) equals 1
The equilibrium is \((R; u(L), u(R))\) which is derived through backward induction. When \(r\) approaches zero, the BEIC is \((L, L; D(L), D(R))\). It is equilibrium iv of the PBE approach, which is ruled out by the intuitive criterion. Note that the BEIC for this limiting case agrees with the equilibrium for the sequential complete and perfect information game, which is represented by Figure 3.8.

The equilibrium is \((L; d(L), d(R))\) which is derived through backward induction.

The above example illustrates the BEIC approach to solving a game of incomplete and perfect information. It also illuminates the relationship between incomplete and perfect information sequential games and complete and perfect information sequential games. When the variance of
type tends to zero, a sequential game with incomplete and perfect information becomes a sequential game with complete and perfect information. The equilibrium of the former should be equal to the equilibrium of the latter in the limiting case. The BEIC approach satisfies this requirement.

For further illustration, the following analysis focuses on the previous game where $0 \leq r < \frac{1}{100}$. The following two Figures show the calculation of the expected values of $x$ and $y$.

![Figure 3.9](image-url) Expected value of $x$
Sequential games with perfect and imperfect information

The process of conjectures is:

Table 3.11  BEIC of the coordination signaling game in Figure 3.6

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Pr (a)</th>
<th>Pr (b)</th>
<th>Pr (x)</th>
<th>Pr (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>U[0,1]</td>
<td>U[0,1]</td>
<td>U[0,1]</td>
<td>U[0,1]</td>
</tr>
<tr>
<td>2</td>
<td>r/10(1−r)</td>
<td>5r/2(1−r)</td>
<td>0.3</td>
<td>0.7</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
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<td>4</td>
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</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

So the BEIC is \((L, L; D(L), D(R))\).10
3.4. Multiple-sided incomplete information sequential games with perfect information

The Bayesian iterative conjecture algorithm is powerful. It could analyze multiple-sided incomplete information sequential games quite easily. The analysis of these games is a formidable, if not impossible, task for the perfect Bayesian equilibrium-based approach. Consider the two-sided incomplete information Beer-Quiche game in Figure 3.11.

The game in Figure 3.11 is an extension of the famous Beer-Quiche game. The left hand side of the game is the original Beer-Quiche game. In the game there are two types of senders, wimpy and surly, and two types of receivers, bully and patrolman. The probability of wimpy is 0.1 and the probability of surly is 0.9. The bully type enjoys picking on the wimpy type. The patrolman type, on the other hand,
has the duty of challenging the surly type when surly orders beer and only if surly orders beer. However, if patrolman challenges wimpy, then patrolman is humiliated. The probability of bully is 0.1 and the probability of patrolman is 0.9.

The perfect Bayesian equilibriums of the above game are:

1. Quiche (wimpy), quiche (surly); no (quiche), no (beer), q1<=1/2; no (quiche), duel (beer), q2<=1/2.
2. Quiche (wimpy), quiche (surly), no (quiche), duel (beer), q1>=1/2; no (quiche), duel (beer) q2<1/2.

The Bayesian iterative conjectures approach solution is presented below:

Let the probability that bully duels when beer is observed to be $u$, the probability that bully duels when quiche is observed to be $v$, the probability that patrolman duels when beer is observed to be $s$ and the probability that patrolman duels when quiche is observed to be $t$. Please note that when quiche is observed, patrolman has the dominant strategy of choosing no duel and therefore $t=0$. Let the probability that surly chooses beer be $x$ and the probability that the wimpy chooses beer be $y$.

Given $u$, $v$, $s$ and $t$, surly chooses beer if

\[
(0.1)[u + 3(1 - u)] + (0.9)[s + 3(1 - s)] > (0.1)2(1 - v) + (0.9)2(1 - t),
\]

is indifferent if

\[
(0.1)[u + 3(1 - u)] + (0.9)[s + 3(1 - s)] = (0.1)2(1 - v) + (0.9)2(1 - t)
\]

and chooses quiche if

\[
(0.1)[u + 3(1 - u)] + (0.9)[s + 3(1 - s)] < (0.1)2(1 - v) + (0.9)2(1 - t).
\]
Given $u$, $v$, $s$ and $t$, wimpy chooses beer if
\[
(0.1)[2(1-u)] + (0.9)[2(1-s)] > (0.1)[v + 3(1-v)] + (0.9)[t + 3(1-t)],
\]
is indifferent if
\[
(0.1)[2(1-u)] + (0.9)[2(1-s)] = (0.1)[v + 3(1-v)] + (0.9)[t + 3(1-t)],
\]
and chooses quiche if
\[
(0.1)[2(1-u)] + (0.9)[2(1-s)] < (0.1)[v + 3(1-v)] + (0.9)[t + 3(1-t)].
\]

When observing beer, bully does not duel if
\[
(-1)(x)\left(\frac{9}{10}\right) + (y)\left(\frac{1}{10}\right) < 0,
\]
is indifferent if
\[
(-1)(x)\left(\frac{9}{10}\right) + (y)\left(\frac{1}{10}\right) = 0,
\]
and duels if
\[
(-1)(x)\left(\frac{9}{10}\right) + (y)\left(\frac{1}{10}\right) > 0.
\]

When observing quiche bully does not duel if
\[
(-1)(1-x)\left(\frac{9}{10}\right) + (1-y)\left(\frac{1}{10}\right) < 0,
\]
is indifferent if
\[
(-1)(1-x)\left(\frac{9}{10}\right) + (1-y)\left(\frac{1}{10}\right) = 0,
\]
and duels if
\[
(-1)(1-x)\left(\frac{9}{10}\right) + (1-y)\left(\frac{1}{10}\right) > 0.
\]
When observing beer patrolman does not duel if
\[(x)\left(\frac{9}{10}\right) + (-1)(y)\left(\frac{1}{10}\right) > 0,\]
is indifferent if
\[(x)\left(\frac{9}{10}\right) + (-1)(y)\left(\frac{1}{10}\right) = 0,\]
and duels if
\[(x)\left(\frac{9}{10}\right) + (-1)(y)\left(\frac{1}{10}\right) < 0.\]

The process therefore converges with \(u=0, v=0, s=1, t=0, x=0\) and \(y=0\). Given the high probability of meeting patrolman, surly chooses quiche. Consequently, wimpy chooses quiche both for impersonating surly to avoid being challenged by bully, and for his intrinsic preference for quiche. Bully chooses not to duel when quiche is observed since the probability of meeting surly is high (0.9).

Now let the probability of bully be 0.9 and the probability of patrolman be 0.1.

The perfect Bayesian equilibriums are:
1. Quiche (wimpy), quiche (surly); no (quiche), duel (beer), \(q_1\geq1/2\); no (quiche), duel (beer), \(q_2\leq1/2\).

<table>
<thead>
<tr>
<th>Order\Pr</th>
<th>u</th>
<th>V</th>
<th>s</th>
<th>t</th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
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<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
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<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
This equilibrium is ruled out by the intuitive criterion.

2. Quiche (wimpy), quiche (surly); no (quiche), duel (beer), \( q_1 \geq 1/2 \); no (quiche), no (beer), \( q_2 \geq 1/2 \).

This equilibrium is ruled out by the intuitive criterion.

3. Beer (wimpy), beer (surly); no (beer), duel (quiche), \( p_1 \geq 1/2 \), no (quiche), duel (beer).

The solution by the Bayesian iterative conjecture approach is:

Given \( u, v, s \) and \( t \), surly chooses beer if

\[
(0.9)\left[ u + 3(1-u) \right] + (0.1)\left[ s + 3(1-s) \right] > (0.9)2(1-v) + (0.1)2(1-t),
\]

is indifferent if

\[
(0.9)\left[ u + 3(1-u) \right] + (0.1)\left[ s + 3(1-s) \right] = (0.9)2(1-v) + (0.1)2(1-t),
\]

and chooses quiche if

\[
(0.9)\left[ u + 3(1-u) \right] + (0.1)\left[ s + 3(1-s) \right] < (0.9)2(1-v) + (0.1)2(1-t).
\]

Given \( u, v, s \) and \( t \), wimpy chooses beer if

\[
(0.9)\left[ 2(1-u) \right] + (0.1)\left[ 2(1-s) \right] > (0.9)\left[ v + 3(1-v) \right] + (0.1)\left[ t + 3(1-t) \right],
\]

is indifferent if

\[
(0.9)\left[ 2(1-u) \right] + (0.1)\left[ 2(1-s) \right] = (0.9)\left[ v + 3(1-v) \right] + (0.1)\left[ t + 3(1-t) \right],
\]
and chooses quiche if

\[ (0.9)[(2−u)] + (0.1)[(2−s)] \leq (0.9)[v + 3(1−v)] + (0.1)[t + 3(1−t)]. \]

The reaction functions for bully and patrolman remain the same and are omitted for the sake of brevity.

The process therefore converges with \( u=0, v=1, s=1, t=0, x=1 \) and \( y=1 \). Given the high probability of meeting bully, surly chooses beer to signal his toughness. Consequently, wimpy chooses beer to impersonate surly to avoid being challenged by bully. Bully chooses not to duel when observing beer since the probability of meeting surly is high (0.9). Needless to say, as the probability of bully increases beyond 0.9 and approaches 1, the same equilibrium holds. That is to say, in the limiting case when the probability of meeting patrolman vanishes, the BEIC solution to the two-sided incomplete information extended Beer-Quiche game agrees with the original Beer-Quiche game. The BEIC solution sounds intuitive and compelling indeed.

As shown in the above example, the Bayesian iterative conjecture approach allows two-sided incomplete information games to be analyzed in a general way that avoids the generation of multiple equilibriums. The example has a

<table>
<thead>
<tr>
<th>Order</th>
<th>Pr</th>
<th>u</th>
<th>v</th>
<th>s</th>
<th>t</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>0.5</td>
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<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
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<td>0</td>
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</tr>
</tbody>
</table>
unique equilibrium and the solution is compellingly insightful and intuitive.

3.5. Conclusions

Is the current concept of rationality in games theory complete without defining rationality in the processing of information and forming of conjectures? The Bayesian iterative conjecture algorithm approach answers no, and attempts to fill in the gap. Bayesian equilibrium by iterative conjectures specifies how rational players in game theoretic situations form and update their conjectures. Rationality in strategies and beliefs is achieved in Bayesian equilibrium by iterative conjectures in a methodical way. It is therefore a useful refinement for sequential games of incomplete information where the beliefs of players need to be specified.

The BEIC approach is good in equilibrium selection. It rules out equilibrium depending upon non-credible threats. In sequential games of incomplete information, it normally narrows down the equilibriums to one and also picks the most compelling equilibrium. Furthermore, the BEIC approach achieves consistency in equilibrium results among different categories of games. When the variance of type tends to zero, a sequential game with incomplete and perfect information becomes a sequential game with complete and perfect information. The equilibrium of the latter must be equal to the equilibrium of the former in the limiting case. The BEIC approach passes this test while the PBE approach fails to. This further confirms the superiority of the Bayesian iterative conjecture algorithm approach.
Notes

2. The example is from Gibbons (1992), p. 176.
3. The example is in Gibbons (1992), p. 181.
4. Refer to Okada (2010), p. 3.
5. Refer to Fudenberg and Tirole (1990), Okada (2010).
8. Refer to Teng (2012a) for a full treatment of the game.
9. Refer to Teng (2012a) for a full treatment of the game.
Simultaneous games

4.1. Introduction

The way the Nash equilibrium approach solves a complete information simultaneous move game is to get the interaction points of the reaction functions. Implicit in this solution algorithm is that there is perfect information and the moves are sequential. That is what a reaction function means: the reaction of one player to the action of another player. That implies perfect information, as you have to observe the actions of the other party before you can react to his actions. If there is simultaneity in moves and the players do not observe the moves of the other players, then they cannot react to the actions of the other players. In this situation, a player decides on his actions according to his conjectures of the actions of the other players. It is clear that conjecture plays a central role here. The reaction functions of a simultaneous game are therefore not really reaction functions as those of a perfect information sequential game, and are best named as virtual reaction functions for differentiation from the real reaction functions of a perfect information sequential game.

In a simultaneous move game, none of the players observe what the other players are doing and they all make their decisions without knowledge of the moves of the other
players and all these are common knowledge. By the very
definition of simultaneous move, even if one of players plays
a particular equilibrium strategy prescribed by the concept
of Nash equilibrium, either a pure strategy or mixed strategy,
the other players still do not observe the actions of that
player. They therefore have to conjecture about the move.
Since what the players think or conjecture will affect their
decisions, it therefore follows that the players have to
conjecture about the other players’ conjectures in their
attempts to predict what the other players are doing or will
be doing.

The BEIC approach could be used to analyze both
simultaneous move games and sequential games. However,
there is one important difference between these two
applications. In sequential games, Bayesian statistical
inferences are conditioned upon the observation of the other
players’ moves, however inaccurate those observations are.
In simultaneous games there is no observation, however
inaccurate, to base the inferences or conjectures about the
moves of the other players on. The player must proceed to
infer or conjecture about the moves of the other players
without any observation. The game has a solution if such
conjectures converge.

By the BEIC approach, a player forms conjectures or
expectations of the actions of the other players, starting with
first order uninformative conjectures and updating the
conjectures using game theoretic and statistical decision
theoretic reasoning. The Bayesian equilibrium by iterative
conjectures, if it exists, is the point of convergence of these
conjectures. Therefore, in a simultaneous game, a BEIC is a
focal point or Schelling point.

Since by the Nash equilibrium approach the players in a
simultaneous game make no conjectures or predictions on
Simultaneous games

the other players’ strategies, the Nash equilibrium or equilibriums could be reached only by chance. This is so because in a simultaneous game a player cannot condition his or her strategy or moves on his or her opponent’s strategy or moves. Alternatively, it has to assume that the players have the default belief that all the other players are playing the particular Nash equilibrium strategy and this is common knowledge. This argument of course is quite problematic for games with multiple Nash equilibriums.

The examples in this chapter illustrate the BEIC solution of complete and incomplete information simultaneous games.

4.2. Complete information simultaneous games

**Investment entry game**

Consider Table 4.1, which is the matrix form representation of an investment entry game.

There are three Nash equilibriums: (w=0, y=1), (w=1, y=0) and (w=1/2, y=1/5). The reaction functions are: w=1 for y<1/5, w ∈ [0,1] for y=1/5 and w=0 for y>1/5; y=1 for w<1/2, y ∈ [0,1] for w=1/2, y=0 for w>1/2.

<table>
<thead>
<tr>
<th>Firm 1</th>
<th>Firm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modern (w)</td>
<td>Enter (y) 0, -2</td>
</tr>
<tr>
<td></td>
<td>Refrain (1-y) 7, 0</td>
</tr>
<tr>
<td>Antique (1-w)</td>
<td>4, 2</td>
</tr>
<tr>
<td></td>
<td>Refrain (1-y) 6, 0</td>
</tr>
</tbody>
</table>
For the investment entry game, the table of conjectures is:

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Probability X</th>
<th>Probability Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>U[0, 1]</td>
<td>U[0, 1]</td>
</tr>
<tr>
<td>2</td>
<td>1/5</td>
<td>1/2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Since the BEIC is a point where conjectures converge, the BEIC approach therefore selects compelling stable equilibrium and eliminates unstable equilibrium (as conjectures will not converge to an unstable equilibrium). As mixed strategy equilibriums are in general unstable, they are generally eliminated in the selection of BEIC.1

**Conventional/symmetrical matching pennies game**

There is no pure strategy Nash equilibrium but only a mixed strategy Nash equilibrium (x=1/2, y=1/2). The reaction functions are x=0 if y>1/2, x ∈ [1,1] if y=1/2, x=1 if y<1/2 and y=1 if x>1/2, y ∈ [0,1] if x=1/2, y=0 if x<1/2.
Starting with \( x \in [0,1] \), then \( E(y) = 1/2 \), and \( x \in [0,1] \) and \( E(x) = 1/2 \) and so on. Starting with \( y \in [0,1] \), then \( E(x) = 1/2 \), and \( y \in [0,1] \) and \( E(y) = 1/2 \) and so on. Therefore, the conjectures do not converge and stay at the uninformative first order conjectures of \( x \in [0,1] \) and \( E(x) = 1/2 \) and \( y \in [0,1] \) and \( E(y) = 1/2 \). This is not surprising, given that it is a zero sum (or pure conflict) game and so there could not be any focus point.

Please note that in this BEIC solution, the players do not intentionally randomize. They are just indifferent between going either way given their conjectures about the other players’ moves. In contrast, the interpretation of the mixed strategy Nash equilibrium of this game is the players randomizing on purpose to keep the other players indifferent between either action.

**Unconventional/asymmetrical matching pennies game**

The following table presents an unconventional/asymmetrical matching pennies game:

There is no pure strategy Nash equilibrium but only mixed strategy Nash equilibrium \( (x=1/2, \ y=2/5) \). The reaction functions are \( x=0 \) if \( y>2/5 \), \( x \in [0,1] \) if \( y=2/5 \), \( x=1 \) if \( y<2/5 \) and \( y=1 \) if \( x>1/2 \), \( y \in [0,1] \) if \( x=1/2 \), \( y=0 \) if \( x<1/2 \).

<table>
<thead>
<tr>
<th>Table 4.4 Unconventional/asymmetrical matching pennies game</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
</tr>
<tr>
<td>Head (x)</td>
</tr>
<tr>
<td>Head (x)</td>
</tr>
<tr>
<td>Tail (1−x)</td>
</tr>
</tbody>
</table>
Starting with $x \in [0,1]$, then $E(y) = 1/2$, and $x = 0$ and $y = 0$ and $x = 1$ and $y = 1$ and so on. Starting with $y \in [0,1]$, then $E(x) = 2/5$, and $y = 0$ and $x = 1$ and $x = 0$ and $y = 0$ and $x = 1$ and so on. The conjectures do not converge and there is no Bayesian equilibrium by iterative conjectures.

The capability of the Bayesian iterative conjecture algorithm to narrow down the number of equilibriums normally to one applies to games with continuous action space as well.

**Dividing cake game (zero sum game)**

There is a delicious cake to be divided between player 1 and player 2. Player 1 and player 2 name their shares $x$ and $y$ simultaneously without observing the action of the other player or communicating with each other. The payoffs are $x$ and $y$ respectively if $x+y \leq 1$ and 0 and 0 if $x+y > 1$.

The reaction function for player 1 is $x = 1 - y$ and the reaction function for player 2 is $y = 1 - x$. The following figure depicts the reaction functions.

![Reaction functions of the dividing cake game](image)
The reaction functions superimpose on each other and therefore all points on the bold line are Nash equilibriums. There are uncountable Nash equilibriums.

The Bayesian iterative conjecture algorithm solution is presented below:

Player 1 solves:

$$\max_x \Pr(x - y \geq 0).$$

With a first order non-informative conjecture on $y$, the above problem becomes:

$$\max_x (1 - x).$$

The first order condition is:

$$1 - 2x = 0$$

The second order condition is satisfied:

$$-2 < 0$$

So the second order conjecture is that $x = \frac{1}{2}$. Player 2 anticipates this and sets $y = \frac{1}{2}$. Player 1 anticipates the conjectures of player 2 and sets $x = \frac{1}{2}$ and the process converges here.

Solving the problem with the other starting point gives the same answer:

Player 2 solves:

$$\max_y \Pr(1 - y \geq x)$$

With a first order non-informative conjecture on $x$, the above problem becomes:

$$\max_y (1 - y)$$

The first order condition is:

$$1 - 2y = 0$$

65
The second order condition is satisfied:

\[-2 < 0\]

So the second order conjecture is that \( y = \frac{1}{2} \). Player 1 anticipates this and sets \( x = \frac{1}{2} \). Player 2 anticipates the conjectures of player 1 and sets \( y = \frac{1}{2} \) and the process converges here.

Both the processes of the conjectures converge at the same point, which is the unique BEIC, \( \left( x = \frac{1}{2}, y = \frac{1}{2} \right) \).

**Cournot competition**

In a product market there are two firms engaging in a Cournot competition, firm 1 and firm 2. Their respective production cost functions are \( C(q_1) = 10q_1 \) and \( C(q_2) = 10q_2 \).

The inverse market demand function is \( p = 210 - \frac{1}{2}(q_1 + q_2) \).

The profit functions of firm 1 and firm 2 are

\[
\pi_1 = \left( 210 - \frac{1}{2}q_1 - \frac{1}{2}q_2 - 10 \right)q_1 \quad \text{and} \quad \pi_2 = \left( 210 - \frac{1}{2}q_2 - \frac{1}{2}q_1 - 10 \right)q_2.
\]

The first order conditions of the profit maximization exercises of firm 1 and firm 2 are:

\[
\frac{\partial \pi_1}{\partial q_1} = 200 - q_1 - \frac{1}{2}q_2 = 0
\]

\[
\frac{\partial \pi_2}{\partial q_2} = 200 - q_2 - \frac{1}{2}q_1 = 0
\]
The second order derivatives are:
\[
\frac{\partial^2 \pi_1}{\partial (q_1)^2} = -1, \quad \frac{\partial^2 \pi_1}{\partial q_1 \partial q_2} = -\frac{1}{2}, \quad \frac{\partial^2 \pi_2}{\partial (q_2)^2} = -1, \quad \frac{\partial^2 \pi_2}{\partial q_1 \partial q_2} = -\frac{1}{2}.
\]

The equilibrium is (locally) strategically stable:
\[
\frac{\partial^2 \pi_1}{\partial (q_1)^2} \cdot \frac{\partial^2 \pi_2}{\partial (q_2)^2} - \frac{\partial^2 \pi_1}{\partial q_1 \partial q_2} \cdot \frac{\partial^2 \pi_2}{\partial q_2 \partial q_1} = \frac{3}{4} > 0.
\]

The reaction functions are \( q_1(q_2) = 200 - \frac{1}{2} q_2 \) and \( q_2(q_1) = 200 - \frac{1}{2} q_1 \), with \( q_1(0) = 200, q_1(400) = 0, q_2(0) = 200 \) and \( q_2(400) = 0 \). The (interior solution) Nash equilibrium is
\[
q_1 = q_2 = \frac{400}{3}, \quad \pi_1 = \pi_2 = \frac{80,000}{9}.
\]

The Bayesian iterative conjecture algorithm approach solution is presented below:

Starting with the first order uninformative conjecture that
\[
q_2 \sim U\left(0, \frac{400}{3}\right), \quad \text{then} \quad E[q_1(q_2)] = \frac{3}{400} \int_0^{400} \left(200 - \frac{1}{2} q_2\right) dq_2 = \frac{500}{3}, \quad \text{then} \quad q_2 \left(q_1 = \frac{500}{3}\right) = \frac{700}{6}, \quad \text{then} \quad q_1 \left(q_2 = \frac{700}{6}\right) = \frac{1700}{12}, \quad \text{and so on, until the process converges at} \quad q_1 = q_2 = \frac{400}{3}.
\]

Given the symmetry of the game, the other conjecture process has the same point of convergence. The BEIC is in this case the same as the unique Nash equilibrium.
4.3 BEIC and refinements of Nash equilibrium

This section compares the BEIC approach with the refining criteria of payoff-dominance, risk-dominance and iterated admissibility. The outstanding result from the comparisons is that the BEIC approach is able to pick the natural focal point of a game when the iterated admissibility criterion fails to.

To begin, consider a game with (weakly) dominated strategies.

Table 4.5 Game with (weakly) dominated strategies

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left (y)</td>
</tr>
<tr>
<td>Up (x)</td>
<td>0, 1</td>
</tr>
<tr>
<td>Down (1-x)</td>
<td>1, 1</td>
</tr>
</tbody>
</table>
Simultaneous games

There are two Nash equilibriums: (D, L) and (U, R). However, strategy U and R are (weakly dominated) strategies for players 1 and 2.

The BEIC solution proceeds as follows: given the first order uninformative conjecture that $x = 0.5$, the second order conjecture is $y = 1$ and the third order conjecture is $x = 0$ and the process converges here. Given the first order uninformative conjecture that $y = 0.5$, the second order conjecture is $x = 0$ and the third order conjecture is $y = 1$ and the process converges here. The unique BEIC is $(x=0, y=1)$ and it agrees with the result from iterative elimination of (weakly) dominated strategies.

**Payoff-dominance and risk dominance**

Consider the following stag hunt game:

<table>
<thead>
<tr>
<th>Table 4.6</th>
<th>BEIC for a game with (weakly) dominated strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order of Conjectures</td>
<td>Probability w</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 4.7</th>
<th>The stag hunt game</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>Stag (y)</td>
</tr>
<tr>
<td>Stag (x)</td>
<td>a, a</td>
</tr>
<tr>
<td>Hare (1-x)</td>
<td>b, 0</td>
</tr>
</tbody>
</table>

$a > b > 0$
There are three Nash equilibriums: (S, S), (H, H) and (x=b/a, y=b/a). Besides the problem of too many Nash equilibriums, the mixed strategy Nash equilibrium is also implausible: as the payoff from playing S increases relative to that of playing H, both players play S with a smaller probability. By the payoff dominance criterion, both players should play S. By the risk dominance criterion, both players should play H.

The Bayesian iterative conjecture algorithm solution is presented below:

For $a>2b$ (and therefore $\frac{a-b}{a} > \frac{1}{2}$), the BEIC solution is:
Simultaneous games

For $a=2b$ (and therefore $\frac{a-b}{a} = \frac{1}{2}$), the BEIC solution is:

**Table 4.8** BEIC for stag hunt game A

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Probability X</th>
<th>Probability Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$U[0,1]$</td>
<td>$U[0,1]$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{a-b}{a}$</td>
<td>$\frac{a-b}{a}$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

For $a<2b$ (and therefore $\frac{a-b}{a} < \frac{1}{2}$), the BEIC solution is:

**Table 4.9** BEIC for stag hunt game B

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Probability X</th>
<th>Probability Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$U[0,1]$</td>
<td>$U[0,1]$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{a-b}{a}$</td>
<td>$\frac{a-b}{a}$</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

**Table 4.10** BEIC for stag hunt game C

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Probability X</th>
<th>Probability Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$U[0,1]$</td>
<td>$U[0,1]$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{a-b}{a}$</td>
<td>$\frac{a-b}{a}$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
By the BEIC approach, when cooperation has more (less) than twice the returns of defect, the players choose cooperation (defect).

A comparison with the payoff-dominance and risk-dominance refinements proposed by Harsanyi and Selten (1988) and Harsanyi (1995) reveals that for the stag hung game, the BEIC approach picks the payoff-dominance equilibrium if the difference between the return from the payoff-dominance equilibrium and the return from the risk-dominance equilibrium is large. If the difference between the returns is small, then the BEIC approach chooses the risk-dominance equilibrium.

**Iterated admissibility**

Iterated admissibility requires that players play only strategies that survive the iterated elimination of (weakly) dominated strategies. The following are two examples that show that the BEIC approach does better than the iterated admissibility criterion.

In the game of Table 4.11, there are two pure strategy Nash equilibriums, (U, L) and (M, C). By the repeated iterative elimination of (weakly) dominated strategies, there is only an equilibrium (U, L).

The BEIC solution:

<table>
<thead>
<tr>
<th>Player 2</th>
<th>L (u)</th>
<th>C (v)</th>
<th>R (1-u-v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U (x)</td>
<td>2, 2</td>
<td>2, 1</td>
<td>1, 1</td>
</tr>
<tr>
<td>M (y)</td>
<td>1, 1</td>
<td>2, 3</td>
<td>3, 2</td>
</tr>
<tr>
<td>D (1-x-y)</td>
<td>1, 4</td>
<td>2, 3</td>
<td>2, 4</td>
</tr>
</tbody>
</table>
Simultaneous games

BEIC gives the same answer as the criterion of iterated admissibility.

In the next example, which is essentially a coordination game, the solutions of the two approaches differ:

The three pure strategy NEs are \((A, a)\) and \((B, b)\) and \((C, c)\). However, \((A, a)\) survives iterated admissibility while \((B, b)\) and \((C, c)\) do not. Furthermore, \((B, b)\) Pareto dominates \((A, a)\) and \((C, c)\) and, \((C, c)\) Pareto dominates \((A, a)\).

Therefore, \((B, b)\) is a natural focal point of the game.

The BEIC solution:

**Table 4.12** BEIC for the game in Table 4.11

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Pr (x)</th>
<th>Pr (y)</th>
<th>Pr (u)</th>
<th>Pr (v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

BEIC gives the same answer as the criterion of iterated admissibility.

In the next example, which is essentially a coordination game, the solutions of the two approaches differ:

The three pure strategy NEs are \((A, a)\) and \((B, b)\) and \((C, c)\). However, \((A, a)\) survives iterated admissibility while \((B, b)\) and \((C, c)\) do not. Furthermore, \((B, b)\) Pareto dominates \((A, a)\) and \((C, c)\) and, \((C, c)\) Pareto dominates \((A, a)\).

Therefore, \((B, b)\) is a natural focal point of the game.

The BEIC solution:

**Table 4.13** A 3-by-3 game

<table>
<thead>
<tr>
<th>Player 2</th>
<th>A(x)</th>
<th>b(y)</th>
<th>c(1-x-y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A(u)</td>
<td>1, 1</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
<tr>
<td>B(v)</td>
<td>0, 10</td>
<td>10, 10</td>
<td>0, 0</td>
</tr>
<tr>
<td>C(1-u-v)</td>
<td>0, 0</td>
<td>9, 2</td>
<td>2, 2</td>
</tr>
</tbody>
</table>

**Table 4.14** BEIC for the game in Table 4.13

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Pr (U)</th>
<th>Pr (V)</th>
<th>Pr (X)</th>
<th>Pr (Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The BEIC approach picks \((B, b)\) as the equilibrium.
4.4 Simultaneous games with incomplete information

The solution of incomplete information simultaneous games proceeds in a likewise manner. Consider the following investment entry game where firm 1, the incumbent, has two types, high investment costs (with probability $a$) or low investment costs (with probability $1-a$).

When the high investment cost firm 1 encounters firm 2 they have the following payoff matrix:

(The complete information simultaneous game with the above payoff matrix has two pure strategy Nash equilibria and a mixed strategy Nash equilibrium, $(w=0, y=1)$, $(w=1, y=0)$ and $(w=1/2, y=1/5)$. The BEIC is $(w=0, y=1)$.)

When low investment cost firm 1 encounters firm 2 they have the following payoff matrix:

(The complete information simultaneous game with the above payoff matrix has two pure strategy Nash equilibria and a mixed strategy Nash equilibrium, $(w=0, y=1)$, $(w=1, y=0)$

---

**Table 4.15** Investment entry game A

<table>
<thead>
<tr>
<th>Firm 1 (High Cost)</th>
<th>Firm 2</th>
<th>Enter ($y$)</th>
<th>Refrain ($1-y$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modern ($w$)</td>
<td>0, -2</td>
<td>7, 0</td>
<td></td>
</tr>
<tr>
<td>Antique ($1-w$)</td>
<td>4, 2</td>
<td>6, 0</td>
<td></td>
</tr>
</tbody>
</table>

**Table 4.16** Investment entry game B

<table>
<thead>
<tr>
<th>Firm 1 (Low Cost)</th>
<th>Firm 2</th>
<th>Enter ($y$)</th>
<th>Refrain ($1-y$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modern ($x$)</td>
<td>3, -2</td>
<td>7, 0</td>
<td></td>
</tr>
<tr>
<td>Antique ($1-x$)</td>
<td>4, 2</td>
<td>6, 0</td>
<td></td>
</tr>
</tbody>
</table>
Simultaneous games

and \( (w=1/2, y=1/2) \). The BEIC is \( (w \in [0,1], (y \in [0,1]) \). The conjectures fail to converge and remain as uninformative conjectures with uniform distribution.)

The Bayesian Nash equilibriums are:

\[
\begin{align*}
& (w = \frac{4a - 2}{4a}, x = 1, \\
& y = \frac{1}{5}, a \geq \frac{1}{2}), \quad (w = 0, x = 1, y \in \left[\frac{1}{5}, \frac{1}{2}\right], a = \frac{1}{2}), \quad (w = 0, \\
& x = \frac{1}{2(1-a)}, y = \frac{1}{2}, a < \frac{1}{2}), \quad (w = 1, x = 1, y = 0, a \in [0,1]), \\
& (w = 0, x = 0, y = 1, a \in [0,1]).
\end{align*}
\]

Besides multiple equilibriums, the set of Bayesian Nash equilibriums also has the following implausible features:

1. When it is more likely to have a high cost firm 1, that is, \( a > 1/2 \), firm 2 enters with a lower probability, that is, \( y = 1/5 \), and when it is more likely to have a low cost firm 1, that is, \( a < 1/2 \), firm 2 enters with a higher probability, that is, \( y = 1/2 \);

2. \( \frac{\partial x}{\partial a} = \frac{2}{4(1-a)^2} > 0 \) for \( a < 1/2 \) and \( x = 1 \) for \( a \geq 1/2 \), that is, as the probability of having a high cost firm 1 becomes greater, the low cost firm 1 plays modern with higher probability.

3. \( w = 0 \) for \( a \leq 1/2 \) and \( \frac{\partial w}{\partial a} = \frac{1}{2a^2} > 0 \) for \( a > 1/2 \), that is, as the probability of having a high cost firm 1 becomes greater, the high cost firm 1 plays modern with higher probability.

The BEIC solution is as follow
Table 4.17  BEIC for the investment entry game

<table>
<thead>
<tr>
<th>Order of Conjectures</th>
<th>Pr (w)</th>
<th>Pr (x)</th>
<th>Pr (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>U[0,1]</td>
<td>U[0,1]</td>
<td>U[0,1]</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

For whatever value of $a$, the BEIC is that both types of firm 1 choose antique and firm 2 always enters.

The reason is as follows: should firm 2 encounter a high cost firm 1 in a game with complete information, the BEIC is that the high cost firm 1 chooses antique, and firm 2 enters. On the other hand, when firm 2 encounters a low cost firm 1 in a game with complete information, the BEIC is that both firms are indifferent between their two possible courses of actions. The BEIC of the first complete information game therefore exercises a dominant influence on the solution of the incomplete information game. Consequently, in the incomplete information game, whatever the type of firm 1, firm 2 always enters and firm 1 always chooses antique, and the BEIC does not depend on the probability of firm 1 being high cost or low cost type.

Given its ability to narrow down the number of equilibrium normally to one, the BEIC approach is useful for solving games with multiple-sided incomplete information, multiple heterogeneous players and multiple decision variables. Solving such games using the current prevailing games theory based upon Nash equilibrium could be a daunting task. There are typically multiple equilibriums. Generally, the game has to be simplified to make analysis possible. For instance, there is
very little game theoretic research with two-sided incomplete information and none with three or more-sided incomplete information. However, real life situations involving strategic interactions with two-sided or multiple-sided incomplete information are very common. Hence, there is a strong need for an approach that could solve such games generally, and generate lesser or even a unique equilibrium. It is to this task that the next sub-section devotes itself. The next section uses the BEIC approach to solve a two-sided incomplete information game. The approach could easily be generalized to three or more-sided incomplete information.

Two-sided incomplete information investment entry game

Consider the following investment entry game. Firm 1 is the incumbent and firm 2 is the potential entrant. Both firm 1 and firm 2 have two types, high investment costs or low investment costs. The probability that firm 1 is of the high cost type is 3/4 and the probability that firm 2 is the high cost type is 1/10. These probabilities are independent of each other and are common knowledge.

When the high investment cost firm 1 faces the low investment cost firm 2 they have the following payoff matrix:

<table>
<thead>
<tr>
<th>Firm 2 (low cost)</th>
<th>Enter (y)</th>
<th>Refrain (1-y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1 (high cost)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modern (x)</td>
<td>0, -2</td>
<td>7, 0</td>
</tr>
<tr>
<td>Antique (1-x)</td>
<td>4, 2</td>
<td>6, 0</td>
</tr>
</tbody>
</table>
When the high cost firm 1 faces the high cost firm 2 they have the following payoff matrix:

**Table 4.19**  High cost firm 1 vs high cost firm 2

<table>
<thead>
<tr>
<th>Firm 1 (high cost)</th>
<th>Firm 2 (high cost)</th>
<th>Enter (y)</th>
<th>Refrain (1-y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modern (w)</td>
<td>0, -5</td>
<td>7, 0</td>
<td></td>
</tr>
<tr>
<td>Antique (1-w)</td>
<td>4, 1</td>
<td>6, 0</td>
<td></td>
</tr>
</tbody>
</table>

If the low investment cost firm 1 encounters the low cost firm 2, then they have the following payoff matrix:

**Table 4.20**  Low cost firm 1 vs low cost firm 2

<table>
<thead>
<tr>
<th>Firm 1 (low cost)</th>
<th>Firm 2 (low cost)</th>
<th>Enter (y)</th>
<th>Refrain (1-y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modern (w)</td>
<td>3, -2</td>
<td>7, 0</td>
<td></td>
</tr>
<tr>
<td>Antique (1-w)</td>
<td>4, 2</td>
<td>6, 0</td>
<td></td>
</tr>
</tbody>
</table>

If the low investment cost firm 1 encounters the high investment cost firm 2, then they have the following payoff matrix:

**Table 4.21**  Low cost firm 1 vs high cost firm 2

<table>
<thead>
<tr>
<th>Firm 1 (low cost)</th>
<th>Firm 2 (high cost)</th>
<th>Enter (y)</th>
<th>Refrain (1-y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modern (w)</td>
<td>3, -5</td>
<td>7, 0</td>
<td></td>
</tr>
<tr>
<td>Antique (1-w)</td>
<td>4, 1</td>
<td>6, 0</td>
<td></td>
</tr>
</tbody>
</table>
The reaction functions of the respective types of firm 1 and firm 2 are:

I. \( w(y, z) = 1 \) if \( 1 > \frac{9}{2}y + \frac{1}{2}z, \) \( w(y, z) \in [0,1] \) if \( 1 = \frac{9}{2}y + \frac{1}{2}z, \)
\[ w(y, z) = 0, \] if \( 1 < \frac{9}{2}y + \frac{1}{2}z \)

II. \( x(y, z) = 1 \) if \( 1 > \frac{9}{5}y + \frac{1}{5}z, \) \( x(y, z) \in [0,1] \) if \( 1 = \frac{9}{5}y + \frac{1}{5}z, \)
\[ x(y, z) = 0, \] if \( 1 < \frac{9}{5}y + \frac{1}{5}z \)

III. \( y(w, x) = 1 \) if \( 1 > \frac{3}{2}w + \frac{1}{2}x, \) \( y(w, x) \in [0,1] \) if \( 1 = \frac{3}{2}w + \frac{1}{2}x, \)
\[ y(w, x) = 0, \] if \( 1 < \frac{3}{2}w + \frac{1}{2}x \)

IV. \( z(w, x) = 1 \) if \( 1 > \frac{9}{2}w + \frac{3}{2}x, \) \( z(w, x) \in [0,1] \) if \( 1 = \frac{9}{2}w + \frac{3}{2}x, \)
\[ z(w, x) = 0, \] if \( 1 < \frac{9}{2}w + \frac{3}{2}x \)

The solution by the Bayesian iterative conjecture algorithm is presented below:

<table>
<thead>
<tr>
<th>Order</th>
<th>Pr</th>
<th>W</th>
<th>x</th>
<th>y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>U[0,1]</td>
<td>U[0,1]</td>
<td>U[0,1]</td>
<td>U[0,1]</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1/6</td>
<td>1/2</td>
<td>1/2</td>
<td>2/27</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
Figure 4.4 \( E(w) \)

\[ 1 = \frac{5}{2} y + \frac{1}{2} z \]

Figure 4.5 \( E(x) \)

\[ 1 = \frac{5}{2} y + \frac{1}{2} z \]
So the Bayesian equilibrium by iterative conjectures is $w=0$, $x=0$, $y=1$, $z=1$. As the chance of meeting the low cost firm 2 is very high, both types of firm 1 choose antique. As the chances of meeting the high cost firm 1 is quite high, both types of firm 2 choose enter.

**Comparative statics**

The comparative statics exercise below illustrates that the BEIC approach generates results that are intuitive and compelling. Now let the probability of firm 1 being the low cost type be $3/4$ and the probability of firm 1 being the high
cost type be 1/4 and the probability of firm 2 being the low cost type be 1/0 and the probability of firm 2 being the high cost type be 9/10.

The reaction functions are:

I. \( w(y, z) = 1 \) if \( 1 > \frac{1}{2} y + \frac{9}{2} z, w(y, z) \in [0,1] \) if \( 1 = \frac{1}{2} y + \frac{9}{2} z, \)

\( w(y, z) = 0, \) if \( 1 < \frac{1}{2} y + \frac{9}{2} z \)

II. \( x(y, z) = 1 \) if \( 1 > \frac{1}{5} y + \frac{9}{5} z, x(y, z) \in [0,1] \) if \( 1 = \frac{1}{5} y + \frac{9}{5} z, \)

\( x(y, z) = 0, \) if \( 1 < \frac{1}{5} y + \frac{9}{5} z \)

III. \( y(w, x) = 1 \) if \( 1 > \frac{1}{2} w + \frac{3}{2} x, y(w, x) \in [0,1] \) if \( 1 = \frac{1}{2} w + \frac{3}{2} x, \)

\( y(w, x) = 0, \) if \( 1 < \frac{1}{2} w + \frac{3}{2} x \)

IV. \( x(w, x) = 1 \) if \( 1 > \frac{3}{2} w + \frac{9}{2} x, x(w, x) \in [0,1] \) if \( 1 = \frac{3}{2} w + \frac{9}{2} x, \)

\( x(w, x) = 0, \) if \( 1 < \frac{3}{2} w + \frac{9}{2} x \)

<table>
<thead>
<tr>
<th>Order</th>
<th>w</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
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<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
So the Bayesian equilibrium by iterative conjectures is \( w=1, x=1, y=0, z=0 \). As the chances of meeting the low cost firm 2 is very low now, both types of firm 1 choose modern. As the chances of meeting the low cost firm 1 is quite high now, both types of firm 2 choose refrain.

### 4.5. Conclusions

There are many refinements of Nash equilibrium and many alternative equilibrium concepts. One line of research focuses on modeling how players in a game conjecture about the strategic choices of other players. Bayesian methodologies are typically involved in this line of research. The examples in this chapter show that the BEIC approach has the merit of narrowing down the equilibriums generally to one. The BEIC selects the most compelling equilibrium.

In general, when there is a unique stable pure strategy Nash equilibrium, that Nash equilibrium is also the BEIC solution. When there are multiple Nash equilibriums (which could be stable or unstable or mixed), or an unstable Nash equilibrium (as in the case of a 2-by-2 matrix game with only mixed strategy equilibrium), then the solutions of the two approaches differ. The BEIC approach typically picks a unique equilibrium that is stable and rules out unstable equilibriums.

### Notes

1. Refer to Aumann (1985) for criticisms of mixed strategy equilibrium.
Conclusions

To conclude, the major differences between the BEIC approach and the current Nash equilibrium-based approach are:

1. A unified solution algorithm
The current Nash equilibrium-based games theory has different equilibrium concepts or refinements, and their associated solution algorithms for different types of games are: Nash equilibrium for complete information simultaneous games, sub-game perfect equilibrium for complete information sequential games, Bayesian Nash equilibrium for incomplete information simultaneous games, perfect Bayesian equilibrium for incomplete information sequential games, and many more. Consequently, there are inconsistencies in solutions from using different equilibrium concepts and solution algorithms when one uses the current Nash equilibrium-based games theory. This is not the case for the Bayesian iterative conjecture approach. It is a unified approach: the same equilibrium concept (Bayesian equilibrium by iterative conjectures) and solution algorithm (Bayesian iterative conjecture algorithm) applies to all the aforementioned types of games and more.
2. Use of reaction functions
The current Nash equilibrium-based games theory solves for equilibriums by constructing reaction functions and looks for their intersections. The BEIC approach constructs reaction functions as well. However, it uses first order uninformative conjectures and reaction functions to derive higher and higher orders of conjectures until a convergence of conjectures is achieved.

3. Definition of rationality
The Nash equilibrium-based approach does not have rationality in the processing of information and forming of conjectures or predictions, that is, rationality in (statistical) decision rule. It deals with the issue of processing information and forming predictions in an ad hoc manner through perfect Bayesian equilibrium and its many refinements. In contrast, rationality in the processing of information and forming of predictions is the very foundation of the BEIC.

4. Equilibrium in strategic space versus equilibrium in subjective probability space
The Nash equilibrium approach defines equilibrium in the strategic/actions space. The incorporation of beliefs in incomplete information games does not change that basic feature. In contrast, the BEIC approach defines equilibrium in subjective probability space through the convergence of conjectures. Of course, for conjectures to converge, they must also be consistent with the equilibrium they support, and so the BEIC’s equilibrium in subjective probability space includes equilibrium in strategic/action space as well.

5. Objective versus subjective probability distribution function
The BEIC is based on the Bayesian view of subjective probability. This allows the tracing of the updating of
conjectures from first order uninformative conjectures to higher and higher orders of conjectures until convergence. The Nash equilibrium-based approach largely sticks to the classical or frequentist view of probability. However, it makes an exception in sequential games of incomplete information with pooling equilibriums by resorting to subjective probability in the specification of off equilibrium beliefs.

Notes

2. Refer to Harsanyi (1982a, 1982b), Kadane (1982a, 1982b) for an intellectual exchange between these two views of probability and games theory.
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